General solutions for multispin two-time correlation and response functions in the Glauber–Ising chain

2004 J. Phys. A: Math. Gen. 37 9

(http://iopscience.iop.org/0305-4470/37/1/002)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 9–49 PII: S0305-4470(04)66007-0

General solutions for multispin two-time correlation and response functions in the Glauber–Ising chain

Peter Mayer and Peter Sollich

Department of Mathematics, King's College, Strand, London, WC2R 2LS, UK

Received 10 July 2003
Published 10 December 2003
Online at stacks.iop.org/JPhysA/37/9 (DOI: 10.1088/0305-4470/37/1/002)

Abstract

The kinetic Glauber–Ising spin chain is one of the very few exactly solvable models of non-equilibrium statistical mechanics. Nevertheless, existing solutions do not yield tractable expressions for two-time correlation and response functions of observables involving products of more than one or two spins. We use a new approach to solve explicitly the full hierarchy of differential equations for the correlation and response functions. From this general solution follow closed expressions for arbitrary multispin two-time correlation and response functions, for the case where the system is quenched from equilibrium at $T_i > 0$ to some arbitrary $T \geqslant 0$. By way of application, we give the results for two- and four-spin two-time correlation and response functions. From the standard mapping, these also imply new exact results for two-time particle correlation and response functions in one-dimensional diffusion limited annihilation.

PACS number: 05.20.−y

Introduction

Glauber did some of his pioneering work in the field of nonequilibrium statistical mechanics when he introduced the kinetic version of the one-dimensional Ising model [1] about 40 years ago. He made a very careful choice for the model dynamics which, while resembling characteristic features of ferromagnets, also admits an analytic approach. Amongst other things Glauber actually derived the general solutions for one- and two-spin correlation functions in his original paper.

Since then the Glauber–Ising model has continuously appeared in the literature in various contexts. For obvious reasons it has been studied in much detail as a toy model for ferromagnets [1–5]. A link to one-dimensional diffusion–reaction processes has extended its relevance further [6–8], and the model has also been employed as an abstract metaphor for tapped granular media [9]. The most recent occurrences in the literature relate to nonequilibrium

fluctuation–dissipation relations and related questions in higher-dimensional Ising models at criticality [10–12].

A large fraction of the publications that deal with the Glauber–Ising chain either make direct use of the results derived by Glauber [1], or use facts which can be obtained directly using his approach. This is rather remarkable, seeing as (at least) three different versions of general solutions for arbitrary order one-time spin correlations have been derived in the mean time. The authors of [2] showed that a particular combination of correlation functions, denoted the C_n-functions, satisfy simpler dynamical equations than the correlations themselves. From the general solution given for the C_n-functions, arbitrary (one-time) correlations could be obtained. As pointed out by the authors, however, their general expressions are rather complicated already for the case of three-spin correlations. In [13], the rigorous general solution of the master equation was obtained via a Fermion mapping. There, arbitrary correlation functions can in principle be found by expanding the associated operator according to a given scheme. However, as pointed out in the paper, for correlations of more than two spins the amount of algebra involved increases very rapidly, and the formalism is not suitable for extracting two-time quantities. Finally, a rigorous and exact solution for the generating function of correlators has been found more recently [14] using Grassmann variables. The formalism is very compact and yields an explicit expression for the generating function. Naturally, correlation functions are encoded implicitly in this result and follow by taking appropriate derivatives, but multispin or two-time quantities are again very difficult to obtain. The domain size distribution after quenching the system from a completely uncorrelated initial state to zero temperature is also briefly discussed in [14]. The latter quantity was first obtained in [15] using yet another method, namely the analogy between random walk problems and zero temperature dynamics of one-dimensional Ising and Potts models. Via this route the authors of [15] also found expressions for one-time correlation functions of arbitrary order, valid however only for the dynamics after a quench from an uncorrelated initial state.

In this paper we derive yet another version of the general solution for arbitrary correlation functions in the finite Glauber–Ising model subject to generic initial conditions. Compared to the existing results, our solution is in the most explicit form and cannot be simplified further for generic initial conditions. Like the authors of [1, 2], we start from the hierarchy of differential equations for the correlation (or response) functions. However, instead of trying to solve the hierarchy directly as a whole, we judiciously split it into subsystems of inhomogeneous differential equations. These subsystems are first solved separately; arbitrary correlation functions then follow in a recursive manner. Crucially, this recursion can be solved explicitly, giving in the end a rather compact solution of the full hierarchy. This result is of interest in itself, but not directly useful for practical purposes since it involves summations over a number of terms growing with the system size N. However, in the bulk of the paper we show that if the system is quenched from equilibrium at some initial temperature, then all sums involving an extensive number of terms can be performed. This gives the desired explicit expressions for arbitrary multispin two-time correlation and response functions, involving sums of only a finite number of terms.

In section 1 we start with a recap of the definition of the finite one-dimensional Glauber–Ising model, and give the hierarchies of differential equations governing the evolution of multispin one- and two-time correlation (and response) functions. The general solution of these hierarchies, all of which have the same structure, is then derived in section 2. We briefly discuss the procedure of taking the thermodynamic limit in our result, which is in fact straightforward, and then proceed with the rigorous analysis of the finite model. First we verify in section 3 that the well-known equilibrium correlations at $T > 0$ are retrieved correctly. From these, one-time multispin correlation functions after a quench from equilibrium at $T_\mathrm{i} > 0$

to arbitrary $T \geqslant 0$ follow, as shown in section 4. Here we make contact with the results given for the infinite chain given in [2], in [15] (infinite chain, $T_\mathrm{i} = \infty$ and $T = 0$) and in [14] (finite chain, $T_\mathrm{i} = \infty$). Using these one-time correlations as initial conditions we finally obtain the two-time multispin correlations (section 5) and response functions (section 6) after a quench; these are the core new results of our study. In section 7, we illustrate the procedure of extracting particular two-time functions from these general results. We briefly discuss the spin–spin correlation and response functions, already known from [10, 11], and then turn to domain-wall correlations and responses. There, as obvious applications, we give exact results and scaling expansions for the time-dependent equilibrium domain-wall autocorrelation at $T > 0$ and for its two-time analogue for the out-of-equilibrium dynamics at $T = 0$. We also briefly discuss the connection between the Glauber–Ising chain and diffusion-limited reaction processes, indicating how, e.g. multi-particle two-time correlations in the latter processes may be obtained from our results. We conclude in section 8 with a summary and discussion of our work. Useful representations and identities for the functions appearing in our calculations are summarized in a number of appendices.

Readers more interested in our results, rather than their derivation, could omit sections 2–6 on a first reading. They need to refer only to the end of sections 5 and 6, where we state explicitly the expressions for two and four-spin two-time correlation and response functions. The applications described in section 7, which are cross-referenced to the relevant sections in the derivations, should then make it clear how our general results are evaluated in practice.

1. Dynamical equations

The finite, one-dimensional Glauber–Ising model comprises a ring of N Ising spins $\sigma_n \in \{-1, +1\}$ that evolve stochastically in time. We denote by $p(\boldsymbol{\sigma}, t)$ the probability to find the system in state $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ at time t. Single spin-flip or heat-bath dynamics are described by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p(\boldsymbol{\sigma}, t) = \sum_{n=1}^{N} [w_n(F_n\boldsymbol{\sigma}) p(F_n\boldsymbol{\sigma}, t) - w_n(\boldsymbol{\sigma}) p(\boldsymbol{\sigma}, t)] \tag{1}$$

where $w_n(\boldsymbol{\sigma})$ is the rate for flipping the spin σ_n in state $\boldsymbol{\sigma}$, and F_n denotes the spin-flip operator $F_n\boldsymbol{\sigma} = (\sigma_1, \ldots, -\sigma_n, \ldots, \sigma_N)$. Glauber's choice $w_n(\boldsymbol{\sigma}) = \frac{1}{2}\big[1 - \frac{\gamma}{2}\sigma_n(\sigma_{n-1} + \sigma_{n+1})\big]$ for the transition rates—with $\gamma = \tanh(2J/T)$—is such that the equilibrium solution of (1) satisfies detailed balance and coincides with the Boltzmann distribution for the Ising Hamiltonian $\mathcal{H} = -J \sum_n \sigma_n\sigma_{n+1}$ at temperature T. The rates may equivalently be written as

$$w_n(\boldsymbol{\sigma}) = \frac{1}{1 + \mathrm{e}^{\Delta_n \mathcal{H}(\boldsymbol{\sigma})/T}} \tag{2}$$

where $\Delta_n\mathcal{H}(\boldsymbol{\sigma}) = \mathcal{H}(F_n\boldsymbol{\sigma}) - \mathcal{H}(\boldsymbol{\sigma})$ is the energy shift caused by flipping the nth spin. We remark that whenever site indices exceed the range $1, 2, \ldots, N$, as occurs for instance in the expression for the rates $w_1(\boldsymbol{\sigma})$ and $w_N(\boldsymbol{\sigma})$, we use N-periodicity of the ring, that is $\sigma_{N+1} = \sigma_1$ and $\sigma_0 = \sigma_N$.

1.1. Correlation functions

We use the notation

$$C_{\boldsymbol{i}}^{(k)}(t) = \big\langle \sigma_{i_1}(t) \cdots \sigma_{i_k}(t) \big\rangle = \sum_{\boldsymbol{\sigma}} \sigma_{i_1} \cdots \sigma_{i_k} p(\boldsymbol{\sigma}, t) \tag{3}$$

for the (disconnected) k-spin correlation function between sites $1 \leqslant i_1, \ldots, i_k \leqslant N$ at time t. The summation $\sum_{\boldsymbol{\sigma}}$ obviously runs over the 2^N states of the system. From the master equation (1) one readily obtains the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t} C_{\boldsymbol{i}}^{(k)}(t) = -k C_{\boldsymbol{i}}^{(k)}(t) + \frac{\gamma}{2} \sum_{\eta=1}^{k} \left(C_{\boldsymbol{i}-\boldsymbol{e}_\eta}^{(k)}(t) + C_{\boldsymbol{i}+\boldsymbol{e}_\eta}^{(k)}(t) \right) \tag{4}$$

provided that the index vector $\boldsymbol{i} = (i_1, \ldots, i_k)$ has pairwise distinct components. The $\boldsymbol{e}_\eta$ appearing in (4) are k-dimensional unit-vectors along the η-direction, that is $\boldsymbol{e}_1 = (1, 0, \ldots, 0), \boldsymbol{e}_2 = (0, 1, \ldots, 0), \ldots, \boldsymbol{e}_k = (0, 0, \ldots, 1)$. Due to the index-shifts $\boldsymbol{i} \pm \boldsymbol{e}_\eta$, (4) induces a set of coupled differential equations over all k-spin correlations. These equations are, however, not closed as for some $\boldsymbol{i}$ the index-shifts create a pair of equal components in $\boldsymbol{i} \pm \boldsymbol{e}_\eta$. In this case, the order k of the correlation function is reduced by two, since $\sigma_n^2 = 1$ for Ising spins; more precisely

$$\exists\, 1 \leqslant \mu < \nu \leqslant k : i_\mu = i_\nu \Rightarrow C_{\boldsymbol{i}}^{(k)}(t) = C_{\boldsymbol{i}\backslash(i_\mu, i_\nu)}^{(k-2)}(t). \tag{5}$$

Here we have introduced the short-hand $\boldsymbol{i}\backslash(i_\mu, i_\nu) = (i_1, \ldots, i_{\mu-1}, i_{\mu+1}, \ldots, i_{\nu-1}, i_{\nu+1}, \ldots, i_k)$ to indicate that the components i_μ, i_ν are to be removed from $\boldsymbol{i}$. Since (4) applies for any $k \geqslant 1$ and $\boldsymbol{i}$ with $1 \leqslant i_1, \ldots, i_k \leqslant N$ pairwise distinct, the dynamics of $(k-2)$-spin correlations are again governed by (4) and in turn link to $(k-4)$-spin correlations, etc. For odd k this hierarchy of differential equations, linked by the hierarchical property (5) of correlation functions, is closed at level $k = 1$ where the formation of index-pairs becomes impossible. The equations on even levels, on the other hand, are closed when an index-pair occurs on level $k = 2$, giving $C^{(0)}(t) = \sum_{\boldsymbol{\sigma}} \sigma_i^2 p(\boldsymbol{\sigma}, t) = \sum_{\boldsymbol{\sigma}} p(\boldsymbol{\sigma}, t) = 1$. Therefore the hierarchy of differential equations defined by (4), (5) over all even (odd) levels k' in the range $0 \leqslant k' \leqslant k$ constitutes a closed set of equations for the evolution of k-spin correlation functions with even (odd) k, respectively.

1.2. Two-time correlations

We define multispin two-time correlation functions by

$$\begin{aligned} C_{\boldsymbol{i},\boldsymbol{j}}^{(k,l)}(t, t') &= \left\langle \sigma_{i_1}(t) \cdots \sigma_{i_k}(t) \sigma_{j_1}(t') \cdots \sigma_{j_l}(t') \right\rangle \\ &= \sum_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} \sigma_{i_1} \cdots \sigma_{i_k} p(\boldsymbol{\sigma}, t | \boldsymbol{\sigma}', t') \sigma'_{j_1} \cdots \sigma'_{j_l} p(\boldsymbol{\sigma}', t'). \end{aligned} \tag{6}$$

Here $p(\boldsymbol{\sigma}, t | \boldsymbol{\sigma}', t')$ denotes the conditional probability to find the system in state $\boldsymbol{\sigma}$ at time $t \geqslant t'$ given it was in state $\boldsymbol{\sigma}'$ at t'. This satisfies the master equation (1),

$$\frac{\partial}{\partial t} p(\boldsymbol{\sigma}, t | \boldsymbol{\sigma}', t') = \sum_{n=1}^{N} [w_n(F_n \boldsymbol{\sigma}) p(F_n \boldsymbol{\sigma}, t | \boldsymbol{\sigma}', t') - w_n(\boldsymbol{\sigma}) p(\boldsymbol{\sigma}, t | \boldsymbol{\sigma}', t')] \tag{7}$$

with the initial condition $p(\boldsymbol{\sigma}, t' | \boldsymbol{\sigma}', t') = \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'}$; here $\delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} = \prod_{n=1}^{N} \delta_{\sigma_n, \sigma'_n}$ is the Kronecker delta for states, which equals 1 for $\boldsymbol{\sigma} = \boldsymbol{\sigma}'$ and vanishes otherwise. We deduce from (7) the evolution equations for two-time correlations

$$\frac{\partial}{\partial t} C_{\boldsymbol{i},\boldsymbol{j}}^{(k,l)}(t, t') = -k C_{\boldsymbol{i},\boldsymbol{j}}^{(k,l)}(t, t') + \frac{\gamma}{2} \sum_{\eta=1}^{k} \left(C_{\boldsymbol{i}-\boldsymbol{e}_\eta,\boldsymbol{j}}^{(k,l)}(t, t') + C_{\boldsymbol{i}+\boldsymbol{e}_\eta,\boldsymbol{j}}^{(k,l)}(t, t') \right) \tag{8}$$

which again hold only for index vectors $\boldsymbol{i}$ with pairwise distinct components. It is clear from the definition that two-time correlations also exhibit hierarchical properties when index-pairs occur in $\boldsymbol{i}$ (or $\boldsymbol{j}$). Therefore (8) again induces a hierarchy of differential equations, linking

(k, l)-spin to $(k-2, l)$-spin correlations, etc. The hierarchies (8) and (4) have the same structure and differ only in the relevant initial values. For the two-time correlations these are

$$C_{\boldsymbol{i},\boldsymbol{j}}^{(k,l)}(t', t') = C_{\boldsymbol{i}\cup\boldsymbol{j}}^{(k+l)}(t'). \tag{9}$$

The abbreviation $\boldsymbol{i}\cup\boldsymbol{j}$ symbolizes that the components of $\boldsymbol{i}$ and $\boldsymbol{j}$ are combined into a new vector $(i_1, \ldots, i_k, j_1, \ldots, j_l)$. The initial values (9) have to hold on each level k of the hierarchy for all $\boldsymbol{i}$ with pairwise distinct components. To avoid trivial reductions in the order l we restrict ourselves to vectors $\boldsymbol{j}$ with pairwise distinct components.

1.3. Two-time response functions

In order to define response functions, we consider a perturbation $\delta\mathcal{H} = -h_B B(\boldsymbol{\sigma})$ to the Hamiltonian, with a (generally time-dependent) field h_B conjugate to the observable $B(\boldsymbol{\sigma})$. The two-time response function $R_{A,B}(t, t')$ measures the response of the observable $\langle A(t)\rangle = \sum_{\boldsymbol{\sigma}} A(\boldsymbol{\sigma}) p(\boldsymbol{\sigma}, t)$ to a field impulse, where $h_B = h$ during a short time interval $[t', t' + \delta t']$ and $h_B = 0$ at all other times. Transition rates in the presence of a perturbation are defined via (2). Regardless of whether the system has been perturbed or not we may write

$$\langle A(t)\rangle = \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} A(\boldsymbol{\sigma}) p(\boldsymbol{\sigma}, t|\boldsymbol{\sigma}', t' + \delta t') p(\boldsymbol{\sigma}', t' + \delta t'). \tag{10}$$

From (7) the conditional probability $p(\boldsymbol{\sigma}, t|\boldsymbol{\sigma}', t' + \delta t')$ is unaffected by the perturbation since the field vanishes for $t > t' + \delta t'$. To find the effect of the field on $p(\boldsymbol{\sigma}', t' + \delta t')$, one expands to linear order in $\delta t'$ to get

$$p(\boldsymbol{\sigma}', t' + \delta t') = p(\boldsymbol{\sigma}', t') + \delta t' \sum_{n=1}^{N} [w_n(F_n\boldsymbol{\sigma}') p(F_n\boldsymbol{\sigma}', t') - w_n(\boldsymbol{\sigma}') p(\boldsymbol{\sigma}', t')]. \tag{11}$$

Inserting this into (10) and taking the difference between the perturbed and unperturbed cases gives the change $\delta\langle A(t)\rangle$. Writing the expansion of the transition rates to linear order in the field as $w_n(\boldsymbol{\sigma}) = w_n(\boldsymbol{\sigma})|_{h_B=0} + h w_n'(\boldsymbol{\sigma})$ then gives for the two-time response function $R_{A,B}(t, t') = T\delta\langle A(t)\rangle/(h\delta t')$

$$R_{A,B}(t, t') = T \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} A(\boldsymbol{\sigma}) p(\boldsymbol{\sigma}, t|\boldsymbol{\sigma}', t') \sum_{n=1}^{N} [w_n'(F_n\boldsymbol{\sigma}') p(F_n\boldsymbol{\sigma}', t') - w_n'(\boldsymbol{\sigma}') p(\boldsymbol{\sigma}', t')]. \tag{12}$$

Here an extra factor of T has been introduced into the definition of the response function, for later convenience. Since the derivation of equation (12) uses only the master equation (1), it holds for any choice of $A(\boldsymbol{\sigma})$, $B(\boldsymbol{\sigma})$ and $w_n(\boldsymbol{\sigma})$. Comparing with (6) one sees that the t-dependence arises in exactly the same way as for a correlation function between $A(t)$ and some observable at time t'. For multispin response functions, where $\langle A(t)\rangle = \langle \sigma_{i_1}(t)\cdots\sigma_{i_k}(t)\rangle$ and $\delta\mathcal{H} = -h_j(t')\sigma_{j_1}\cdots\sigma_{j_l}$, one thus immediately has the dynamical equations

$$\frac{\partial}{\partial t} R_{\boldsymbol{i},\boldsymbol{j}}^{(k,l)}(t, t') = -k R_{\boldsymbol{i},\boldsymbol{j}}^{(k,l)}(t, t') + \frac{\gamma}{2} \sum_{\eta=1}^{k} \left(R_{\boldsymbol{i}-\boldsymbol{e}_\eta,\boldsymbol{j}}^{(k,l)}(t, t') + R_{\boldsymbol{i}+\boldsymbol{e}_\eta,\boldsymbol{j}}^{(k,l)}(t, t') \right). \tag{13}$$

These are to be interpreted in exactly the same manner as (4) and (8). To avoid a trivial reduction of the order l of the perturbation we require, in analogy with two-time correlations, pairwise distinct components in $\boldsymbol{j}$. The initial conditions for (13) at $t = t'$ also follow from (12): working out $w_n'(\boldsymbol{\sigma})$ from (2) gives

$$w_n'(\boldsymbol{\sigma}) = -\frac{1}{2T} \left[1 - \frac{\gamma^2}{2}(1 + \sigma_{n-1}\sigma_{n+1}) \right] \sigma_{j_1}\cdots\sigma_{j_l} \sum_{\nu=1}^{l} \delta_{n,j_\nu}. \tag{14}$$

Substituting into (12), using $p(\boldsymbol{\sigma}, t'|\boldsymbol{\sigma}', t') = \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}$ and rearranging terms we then find

$$R^{(k,l)}_{\boldsymbol{i},\boldsymbol{j}}(t', t') = \sum_{\mu=1}^{k}\sum_{\nu=1}^{l}\delta_{i_\mu,j_\nu}\left[\left(1-\frac{\gamma^2}{2}\right)C^{(k+l)}_{\boldsymbol{i}\cup\boldsymbol{j}}(t') - \frac{\gamma^2}{2}C^{(k+l+2)}_{\boldsymbol{i}\cup\boldsymbol{j}^\nu}(t')\right]. \tag{15}$$

In (15), $\boldsymbol{j}^\nu = (j_1, \ldots, j_{\nu-1}, j_\nu - 1, j_\nu, j_\nu + 1, j_{\nu+1}, \ldots, j_l)$ is the vector $\boldsymbol{j}$ with the additional components $j_\nu - 1$ and $j_\nu + 1$. The structure of (15) makes sense intuitively: an instantaneous response only occurs if the 'observed' sites $i_i, \ldots, i_k$ overlap with the 'perturbed' sites $j_1, \ldots, j_l$. At level $k = 0$, the sum over μ in (15) is empty; as is done conventionally, we define such empty sums as zero throughout. This makes sense: for $k = 0$, the observable is just a constant, $A(\boldsymbol{\sigma}) = 1$, and its value is unaffected by any perturbation, giving a vanishing response. Equations (4) and (8) are to be read in the same way for $k = 0$.

2. General solution

We have seen in section 1 that the dynamical equations describing the evolution of multispin one- and two-time correlation and two-time response functions all have the same structure. Hence we obtain the various quantities from the same set of differential equations, e.g. (4), by using the appropriate initial conditions. For this reason and in order to unify the analysis we use in this section the symbol $F^{(k)}$ for the dynamical quantity and $A^{(k)}$ for its initial conditions. For the sake of readability we omit indices $\boldsymbol{i}$ in the text except when they are needed for clarity.

All dynamical functions $F^{(k)}$ discussed above are by definition N-periodic in the site indices $i_1, \ldots, i_k$ and invariant under permutations of the latter. But it turns out that these symmetries are not directly helpful for solving the hierarchy (4). We therefore restrict the problem to functions $F^{(k)}$ with ordered indices $\boldsymbol{i} \in N(k)$ where

$$N(k) = \{\boldsymbol{i} \in \mathbb{N}^k \mid 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant N\}. \tag{16}$$

The set $N(k)$ is not closed under the index shifts $\boldsymbol{i} \mapsto \boldsymbol{i} - \boldsymbol{e}_\eta$ and $\boldsymbol{i} \mapsto \boldsymbol{i} + \boldsymbol{e}_\eta$ occurring on the rhs of (4), as shifted components might exceed the range $1, \ldots, N$ or form pairs. Using N-periodicity, permutational symmetry and the hierarchical property, however, each function $F^{(k)}$ with a non-ordered index $\boldsymbol{i} \notin N(k)$ can be expressed in terms of another one, $F^{(k')}$, having an ordered index $\boldsymbol{i}' \in N(k')$. Hence we may formally rewrite (4) on all levels $2 \leqslant k \leqslant N$ as

$$\forall \boldsymbol{i} \in N(k) : \frac{\mathrm{d}}{\mathrm{d}t}F^{(k)}_{\boldsymbol{i}}(t) = \sum_{\boldsymbol{j}\in N(k)} a^{(k)}_{\boldsymbol{i},\boldsymbol{j}}F^{(k)}_{\boldsymbol{j}}(t) + \sum_{\boldsymbol{j}'\in N(k-2)} b^{(k)}_{\boldsymbol{i},\boldsymbol{j}'}F^{(k-2)}_{\boldsymbol{j}'}(t). \tag{17}$$

Note that (17) explicitly separates the terms on the rhs of (4) that link within level k of the hierarchy from those connecting to level $k - 2$ via the hierarchical property. Clearly the first term on the rhs of (4), linking within level k for all $\boldsymbol{i} \in N(k)$, produces diagonal entries in $a^{(k)}$. Also, each function with an index $\boldsymbol{i} - \boldsymbol{e}_\eta \in N(k)$ and $\boldsymbol{i} + \boldsymbol{e}_\eta \in N(k)$ on the rhs of (4) yields corresponding off-diagonal entries in $a^{(k)}$. But when a shifted index is not contained in $N(k)$ several cases have to be distinguished. Here we only discuss the shifts $\boldsymbol{i} - \boldsymbol{e}_\eta$ as a similar reasoning applies to $\boldsymbol{i} + \boldsymbol{e}_\eta$. If $\boldsymbol{i} - \boldsymbol{e}_\eta \notin N(k)$ and $2 \leqslant \eta \leqslant k$ the shift has created an index pair $i_{\eta-1} = i_\eta - 1$. Hence the hierarchical property applies and the corresponding term on the rhs of (4) links to level $k - 2$. Such terms are accounted for in $b^{(k)}$. On the other hand, if $\boldsymbol{i} - \boldsymbol{e}_\eta \notin N(k)$ and $\eta = 1$ then $i_1 = 1$. N-periodicity and permutational symmetry of the underlying dynamical function $F^{(k)}$ allow us to replace the index $\boldsymbol{i} - \boldsymbol{e}_1 = (0, i_2, \ldots, i_k)$ by $(N, i_2, \ldots, i_k)$ and then by $(i_2, \ldots, i_k, N)$, respectively. If $(i_2, \ldots, i_k, N) \in N(k)$ there is a link within level k on the rhs of (4) and we add a corresponding entry to $a^{(k)}$ in the appropriate place $\boldsymbol{i} = (1, i_2, \ldots, i_k)$, $\boldsymbol{j} = (i_2, \ldots, i_k, N)$. Otherwise $i_k = N$ is an index pair and this term

produces an entry in $b^{(k)}$. An explicit representation of $a^{(k)}$ as obtained by this construction is not necessary for our analysis; $b^{(k)}$ is given in (A2) for later reference.

Rewriting (4) in the form (17) obviously does not change the basic problem of having to solve a hierarchy of differential equations. But (17) suggests a somewhat unconventional approach for finding $F^{(k)}$: instead of simultaneously solving the full hierarchy induced by (17) we can focus on a given level $k \geqslant 2$ and consider (17) as an *inhomogeneous* system of differential equations in $F^{(k)}$. Hence we may write

$$F_{\boldsymbol{i}}^{(k)}(t) = \Phi_{\boldsymbol{i}}^{(k)}(t) + \int_0^t \mathrm{d}\tau \sum_{\boldsymbol{j}\in N(k)} \sum_{\boldsymbol{j}'\in N(k-2)} G_{\boldsymbol{i},\boldsymbol{j}}^{(k)}(t-\tau) b_{\boldsymbol{j},\boldsymbol{j}'}^{(k)} F_{\boldsymbol{j}'}^{(k-2)}(\tau) \tag{18}$$

where $\Phi_{\boldsymbol{i}}^{(k)}(t)$ is the solution of the homogeneous equation

$$\forall \boldsymbol{i} \in N(k) : \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\boldsymbol{i}}^{(k)}(t) = \sum_{\boldsymbol{j}\in N(k)} a_{\boldsymbol{i},\boldsymbol{j}}^{(k)} \Phi_{\boldsymbol{j}}^{(k)}(t) \tag{19}$$

satisfying the initial conditions $\forall \boldsymbol{i} \in N(k) : \Phi_{\boldsymbol{i}}^{(k)}(t) = A_{\boldsymbol{i}}^{(k)}$. The second term in (18), on the other hand, is a particular solution of (17) obtained via the Green function $G^{(k)}$ of (19). Thus we may first focus on solving the homogeneous equation (19) on the generic level k, obtaining the Green function on the way. Then (18) constitutes a recursion formula that allows us to successively express any solution on level k in terms of solutions on lower levels $k-2$, $k-4$, etc. The recursion terminates either on level 0 or 1 of the hierarchy where the solutions are known: according to the discussion in section 1, for $k = 0$ we actually have that $F^{(0)}(t) = A^{(0)}$ is constant in time. On level $k = 1$, on the other hand, (4) is homogeneous since no index pairs can be formed in a one-dimensional index vector. So $F^{(1)}(t) = \Phi^{(1)}(t)$ is just the solution of (19) on level $k = 1$.

Note that the applicability of this approach essentially relies on the fact that links between different levels of the hierarchy are unidirectional, i.e. only from k to $k-2$. This is a peculiarity of the one-dimensional Glauber–Ising model.

2.1. Homogeneous solution

Solving the homogeneous equation (19) over ordered indices $\boldsymbol{i} \in N(k)$ is a non-trivial problem. In order to make progress we apply the method of images, in this context already used by Glauber [1], to extend (19) over index vectors $\boldsymbol{i} \in \{1,\ldots,N\}^k$. This procedure is purely for mathematical convenience and only possible because we have reformulated (4) over ordered $\boldsymbol{i} \in N(k)$ in the first place. As a generalization of (16) we introduce the set of index vectors $N_\pi(k)$ that are ordered when permuted by π

$$N_\pi(k) = \{\boldsymbol{i} \in \mathbb{N}^k \mid 1 \leqslant i_{\pi(1)} < i_{\pi(2)} < \cdots < i_{\pi(k)} \leqslant N\}. \tag{20}$$

Here, $\pi \in \mathcal{S}(k)$ denotes a permutation on $\{1,2,\ldots,k\}$ and $\mathcal{S}(k)$ the set of all such permutations. By $(-1)^\pi$ we refer to the sign of a permutation, that is $(-1)^p$ where p is the number of transpositions necessary to perform π. Now we define the homogeneous solutions over $\{1,2,\ldots,N\}^k$ by filling the hypercube with permutationally *antisymmetric* images according to

$$\forall \boldsymbol{i} \in \{1,\ldots,N\}^k : \bar{\Phi}_{\boldsymbol{i}}^{(k)}(t) = \begin{cases} (-1)^\pi \Phi^{(k)}_{(i_{\pi(1)},\ldots,i_{\pi(k)})}(t) & \text{if} \quad \exists \pi \in \mathcal{S}(k) : \boldsymbol{i} \in N_\pi(k) \\ 0 & \text{otherwise} \end{cases}. \tag{21}$$

This equation states that for ordered indices $\boldsymbol{i} \in N(k)$ we have $\bar{\Phi}^{(k)} = \Phi^{(k)}$. Furthermore, when $\boldsymbol{i}$ has pairwise distinct components any transposition in $\boldsymbol{i}$ just changes the sign of $\bar{\Phi}^{(k)}$. Therefore, and since $\bar{\Phi}^{(k)} = 0$ anyway for non-pairwise distinct indices, permuting the index $\boldsymbol{i}$ by $\pi_0 \in S(k)$ changes the sign of $\bar{\Phi}^{(k)}$ by $(-1)^{\pi_0}$ for any $\boldsymbol{i} \in \{1, \ldots, N\}^k$. Setting $t = 0$ in (21) defines the initial conditions $\bar{A}^{(k)}$ over the hypercube. As we explain below $\bar{\Phi}^{(k)}$ satisfies

$$\forall \boldsymbol{i} \in \{1, \ldots, N\}^k : \frac{\mathrm{d}}{\mathrm{d}t} \bar{\Phi}^{(k)}_{\boldsymbol{i}}(t) = -k \bar{\Phi}^{(k)}_{\boldsymbol{i}}(t) + \frac{\gamma}{2} \sum_{\eta=1}^{k} \left(\bar{\Phi}^{(k)}_{\boldsymbol{i}-\boldsymbol{e}_\eta}(t) + \bar{\Phi}^{(k)}_{\boldsymbol{i}+\boldsymbol{e}_\eta}(t) \right) \tag{22}$$

if the functions $\bar{\Phi}^{(k)}$ are moreover extended N-(anti)periodically in each component of $\boldsymbol{i}$ for (even) odd k. We use the term N-antiperiodic in the sense that shifting any component of $\boldsymbol{i}$ by $\pm N$ changes the sign of $\bar{\Phi}^{(k)}$. Note that in contrast to (4), which only holds for index vectors $\boldsymbol{i}$ with pairwise distinct components, (22) applies for all $\boldsymbol{i} \in \{1, \ldots, N\}^k$.

As a first step in proving that $\bar{\Phi}^{(k)}$ satisfies (22) we consider ordered indices $\boldsymbol{i} \in N(k)$. According to (21) we have $\bar{\Phi}^{(k)} = \Phi^{(k)}$ for such $\boldsymbol{i}$. Hence (22) should reproduce the homogeneous equation (19). Equivalently, by expressing $\bar{\Phi}^{(k)}$ in terms of $\Phi^{(k)}$ on the rhs of (22) we must recover the matrix $a^{(k)}$. In the first term on the rhs of (22), and in all terms with shifted indices $\boldsymbol{i} - \boldsymbol{e}_\eta \in N(k)$ and $\boldsymbol{i} + \boldsymbol{e}_\eta \in N(k)$, we have $\bar{\Phi}^{(k)} = \Phi^{(k)}$. So these terms directly reproduce the matrix entries discussed in the text below (17). Now we focus on the case where $\boldsymbol{i} - \boldsymbol{e}_\eta \notin N(k)$; as before, $\boldsymbol{i} + \boldsymbol{e}_\eta \notin N(k)$ is covered by a similar reasoning. If $\boldsymbol{i} - \boldsymbol{e}_\eta \notin N(k)$ and $2 \leqslant \eta \leqslant k$ the shift has created an index pair $i_{\eta-1} = i_\eta - 1$. According to (21) the corresponding $\bar{\Phi}^{(k)}$ is zero as required (case (a) in figure 1). Alternatively, when $\boldsymbol{i} - \boldsymbol{e}_\eta \notin N(k)$ and $\eta = 1$ we must have $i_1 = 1$ and therefore $\boldsymbol{i} - \boldsymbol{e}_1 = (0, i_2, \ldots, i_k) \notin \{1, \ldots, N\}^k$. Using N-(anti)periodicity we replace this index by $(N, i_2, \ldots, i_k) \in \{1, \ldots, N\}^k$ and modify the sign of $\bar{\Phi}^{(k)}$ by $(-1)^{k-1}$. Next, applying a cyclic permutation π_c turns this index into $(i_2, \ldots, i_k, N)$ and changes the sign of $\bar{\Phi}^{(k)}$ according to (21) once more, by $(-1)^{\pi_\mathrm{c}}$. But $(-1)^{\pi_\mathrm{c}} = (-1)^{k-1}$ in fact cancels the sign-change caused by N-(anti)periodicity. So, all in all, if $(i_2, \ldots, i_k, N) \in N(k)$ we recover the correct matrix element of $a^{(k)}$ between $\boldsymbol{i} = (1, i_2, \ldots, i_k)$, $\boldsymbol{j} = (i_2, \ldots, i_k, N)$ (case (b) in figure 1). Otherwise $i_k = N$ and hence $\bar{\Phi}^{(k)} = 0$ as required.

Next consider indices $\boldsymbol{i} \in N_\pi(k)$ for some given $\pi \in S(k)$ that are ordered when permuted by π. Here we use the fact that $\bar{\Phi}^{(k)}$ is antisymmetric under permutations for all $\boldsymbol{i} \in \{1, \ldots, N\}^k$. So, regardless of whether a shifted index on the rhs of (22) is also contained in $N_\pi(k)$ or not, we permute it by π and change the sign of $\bar{\Phi}^{(k)}$ by $(-1)^\pi$. In cases where a shifted index lies outside $\{1, \ldots, N\}^k$ we first use N-(anti)periodicity, then permute by π and finally apply π_c. As discussed above, this again just gives a sign change $(-1)^\pi$. Due to linearity of (22) the overall factor of $(-1)^\pi$ drops out. Therefore π maps each sector $\boldsymbol{i} \in N_\pi(k)$ of (22) onto $\boldsymbol{i} \in N(k)$. Since (22) holds on $\boldsymbol{i} \in N(k)$ this implies that (22) is also satisfied over $\boldsymbol{i} \in N_\pi(k)$ for any $\pi \in S(k)$ (cases (a',b') in figure 1).

We complete the proof that $\bar{\Phi}^{(k)}$ satisfies (22) over $\boldsymbol{i} \in \{1, \ldots, N\}^k$ by showing that it holds for non-pairwise distinct indices, too. In this case there must be at least one index pair, say $i_\mu = i_\nu$, in $\boldsymbol{i}$. Thus the lhs of (22), the first term on the rhs and each term with a shifted index $\eta \neq \mu, \nu$ contain this pair and are identically zero according to (21). We also know that swapping the μth and νth component of $\boldsymbol{i} - \boldsymbol{e}_\mu$ or $\boldsymbol{i} + \boldsymbol{e}_\mu$ turns it into $\boldsymbol{i} - \boldsymbol{e}_\nu$ or $\boldsymbol{i} + \boldsymbol{e}_\nu$, respectively, since $i_\mu = i_\nu$. But it takes an odd number of transpositions to swap index-components, consequently the associated $\bar{\Phi}^{(k)}$ have opposite signs and cancel each other in (22) (case (c) in figure 1). When index shifts exceed the hypercube $\{1, \ldots, N\}^k$ we use N-(anti)periodicity and the same argument applies.

Summarizing so far, by appropriately extending the definition of $\Phi^{(k)}$ beyond the physically relevant range $\boldsymbol{i} \in N(k)$ we have managed to rewrite the homogeneous

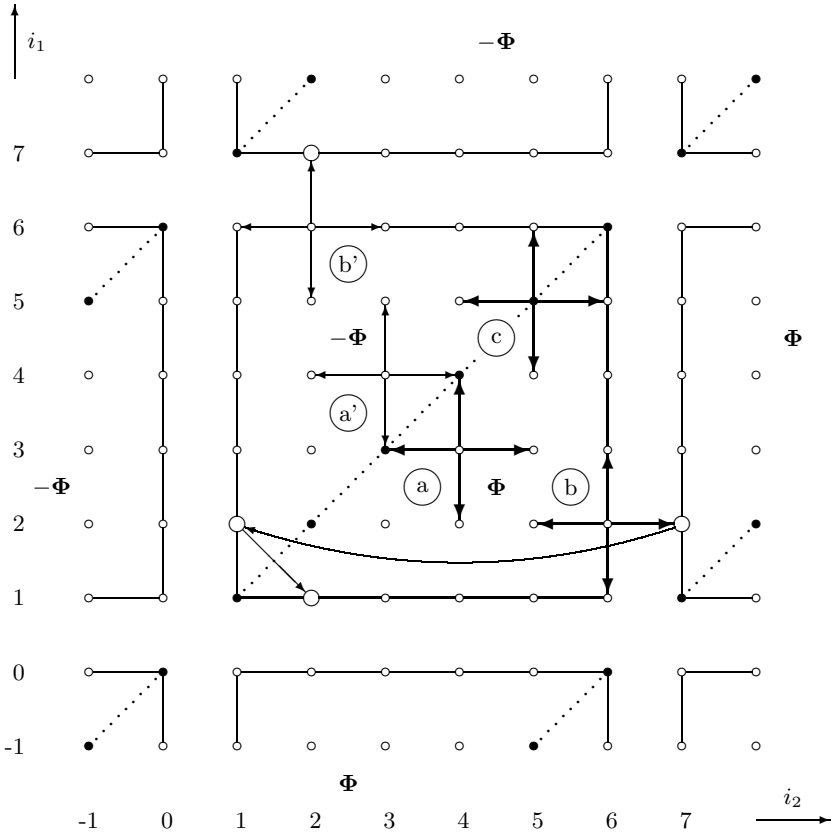

Figure 1. Illustration of the domain extension for homogeneous solutions in a system of size $N = 6$ and at level $k = 2$ of the hierarchy. The square in the centre of the figure represents the region $\{1, \ldots, 6\}^2$ consisting of $N(2)$ (lower triangle) and its image (upper triangle). Associated with each circle in the lower triangle is a homogeneous solution $\bar{\Phi} = \Phi$. The upper triangle contains the permutationally antisymmetric image $\bar{\Phi} = -\Phi$. Along the dotted diagonal the index vector $\boldsymbol{i} = (i_1, i_2)$ is not pairwise distinct and hence $\bar{\Phi} = 0$ (full circles). The surrounding sections of squares represent the N-antiperiodic extension of $\bar{\Phi}$ over $\mathbb{Z}^2$. Finally, the cross of arrows over a circle in position $\boldsymbol{i}$ points to the shifted index vectors $\boldsymbol{i} \pm \boldsymbol{e}_1$ and $\boldsymbol{i} \pm \boldsymbol{e}_2$ in correspondence with the terms on the rhs of (22). Based on this simple example we consider the following special cases discussed in the text: (a) for $\boldsymbol{i} = (3, 4) \in N(2)$ the shifted index vectors $\boldsymbol{i} + \boldsymbol{e}_1 = (4, 4)$ and $\boldsymbol{i} - \boldsymbol{e}_2 = (3, 3)$ are not in $N(2)$ and must therefore contain an index pair (dots on the diagonal). The associated $\bar{\Phi}$ are zero and do not contribute in (22). (b) For $\boldsymbol{i} = (2, 6) \in N(2)$ we have $\boldsymbol{i} + \boldsymbol{e}_2 = (2, 7) \notin N(2)$ (big circle). Using N-antiperiodicity we replace $\bar{\Phi}_{2,7}$ by $-\bar{\Phi}_{2,1}$ which in turn satisfies $\bar{\Phi}_{2,1} = -\Phi_{1,2}$. Hence, this arrow effectively points to $(1, 2) \in N(2)$ and produces the correct term on the rhs of (22). (a',b') If (22) satisfies the cases (a) and (b) in $N(2)$ it is immediately clear that it also holds for (a') and (b') in the image of $N(2)$. (c) For $\boldsymbol{i} = (5, 5)$ we have, e.g. $\boldsymbol{i} - \boldsymbol{e}_1 = (4, 5)$ and $\boldsymbol{i} - \boldsymbol{e}_2 = (5, 4)$. Since $\bar{\Phi}_{5,4} = -\Phi_{4,5}$ these terms cancel in (22). Based on this illustration it is straightforward to prove cases such as $\boldsymbol{i} = (6, 6)$ that are not discussed in detail in the text.

equation (19) in the much simpler form (22). A comparison of (22) with the original hierarchy (4) shows that *antisymmetry* has naturally eliminated all links between different levels k of the hierarchy. Now we can solve (22) over the space of N-(anti)-periodic functions. For this purpose we introduce the discrete Fourier transform $f_q = \mathcal{F}\{f_n\}$ as

$$\forall q \in Q : f_q = \sum_{n=1}^{N} f_n \, \mathrm{e}^{-\mathrm{i}nq} \tag{23}$$

$$\forall n \in \mathbb{Z} : f_n = \frac{1}{N} \sum_{q \in Q} f_q \, \mathrm{e}^{+\mathrm{i}nq}. \tag{24}$$

If one restricts $1 \leqslant n \leqslant N$ in (24), then (23), (24) form a transform pair when Q is any equidistant N-partitioning of the interval $[0, 2\pi]$. But in order to retrieve N-periodic or N-antiperiodic functions from (24) the particular choice $Q = Q_\mathrm{o}$ or $Q = Q_\mathrm{e}$ has to be made, respectively, where

$$Q_\mathrm{e} = \left\{ \frac{\pi}{N} + \frac{2\pi}{N} n \,\middle|\, n = 0, \ldots, N-1 \right\} = \left\{ \frac{\pi}{N}, \frac{3\pi}{N}, \ldots, \frac{(2N-1)\pi}{N} \right\} \tag{25}$$

$$Q_\mathrm{o} = \left\{ \frac{2\pi}{N} n \,\middle|\, n = 0, \ldots, N-1 \right\} = \left\{ 0, \frac{2\pi}{N}, \ldots, \frac{(2N-2)\pi}{N} \right\}. \tag{26}$$

So on even levels k of the hierarchy we set $Q = Q_\mathrm{e}$ as we are dealing with N-antiperiodic functions while $Q = Q_\mathrm{o}$ on odd levels corresponds to N-periodic functions. In either case transforms (23), (24) satisfy the usual shifting and convolution properties of Fourier transforms. Hence the k-fold transform of (22) follows immediately as

$$\forall \boldsymbol{q} \in Q^k : \frac{\mathrm{d}}{\mathrm{d}t} \bar{\Phi}^{(k)}_{\boldsymbol{q}}(t) = \bar{\Phi}^{(k)}_{\boldsymbol{q}}(t) \left(-k + \gamma \sum_{\eta=1}^{k} \cos q_\eta \right). \tag{27}$$

Here $\boldsymbol{q} = (q_1, \ldots, q_k)$ are the Fourier variables corresponding to $\boldsymbol{i} = (i_1, \ldots, i_k)$. Note that we use the subscript to distinguish $\bar{\Phi}^{(k)}_{\boldsymbol{i}}$ from its transform $\bar{\Phi}^{(k)}_{\boldsymbol{q}}$. In Fourier space the evolution equations (27) for $\bar{\Phi}^{(k)}$ are decoupled and can therefore be integrated easily. Then, by inverting the Fourier transform and using the initial condition $\bar{\Phi}^{(k)}_{\boldsymbol{i}}(0) = \bar{A}^{(k)}_{\boldsymbol{i}}$ we obtain

$$\bar{\Phi}^{(k)}_{\boldsymbol{i}}(t) = \sum_{j_1, \ldots, j_k} \prod_{\eta=1}^{k} \mathrm{e}^{-t} I_{i_\eta - j_\eta}(\gamma t) \bar{A}^{(k)}_{\boldsymbol{j}} \tag{28}$$

where the sum runs over $\boldsymbol{j} \in \{1, \ldots, N\}^k$. An expression for $I_n(x)$ is given in (B1) in appendix B, where we also discuss the properties of $I_n(x)$ that are relevant for our analysis. Finally, by expressing the initial conditions $\bar{A}^{(k)}$ in terms of $A^{(k)}$ via (21) and after some rearranging, we obtain the solution $\Phi^{(k)}$ of the homogeneous equation (19) on level $1 \leqslant k \leqslant N$

$$\forall \boldsymbol{i} \in N(k) : \Phi^{(k)}_{\boldsymbol{i}}(t) = \sum_{\boldsymbol{j} \in N(k)} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}}(t) A^{(k)}_{\boldsymbol{j}} \tag{29}$$

with the Green function

$$\forall t > 0 : G^{(k)}_{\boldsymbol{i},\boldsymbol{j}}(t) = \sum_{\pi \in \mathcal{S}(k)} (-1)^\pi \prod_{\eta=1}^{k} \mathrm{e}^{-t} I_{i_\eta - j_{\pi(\eta)}}(\gamma t). \tag{30}$$

The Green function as such is defined over the index range $\boldsymbol{i}, \boldsymbol{j} \in N(k)$. Expression (30), however, carries the antisymmetry of the images and is therefore well defined over $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^k$. For later convenience we note that (30) is in fact permutationally antisymmetric in $\boldsymbol{i}$ and $\boldsymbol{j}$. To see the antisymmetry in $\boldsymbol{j}$, consider the sum over $\pi \in \mathcal{S}(k)$. If, instead of $\boldsymbol{j}$, we substitute a version permuted by π_0, we may use $\pi \circ \pi_0 \in \mathcal{S}(k)$ as the summation variable since the permutations form a group. Hence the replacement of $\boldsymbol{j}$ by $(j_{\pi_0(1)}, \ldots, j_{\pi_0(k)})$ just changes the sign of the Green function by $(-1)^{\pi_0}$, proving permutational antisymmetry. Antisymmetry over $\boldsymbol{i}$ can be shown similarly and is an immediate consequence of the images.

2.2. The recursion formula

In this section we use the recursion formula (18) to construct the general solution of the hierarchy (17). It is convenient to express the homogeneous solution $\Phi^{(k)}$ contained in (18) in terms of the Green function $G^{(k)}$ via (29). This gives

$$F_i^{(k)}(t) = \sum_{j\in N(k)} G_{i,j}^{(k)}(t)A_j^{(k)} + \int_0^t \mathrm{d}\tau \sum_{j\in N(k)} \sum_{j'\in N(k-2)} G_{i,j}^{(k)}(t-\tau)b_{j,j'}^{(k)}F_{j'}^{(k-2)}(\tau). \tag{31}$$

Now we show that a particular sequence of simplifications, essentially based on the similarity of different levels of the hierarchy, can be applied when (31) is iterated. Our proof is inductive, hence we start with proposing a structure for the result. Working out the first few iterations of (31) suggests that $L+1 \leqslant \lfloor \frac{k}{2} \rfloor$ iterations should give the following expression:

$$F_i^{(k)}(t) = \sum_{l=0}^{L} \sum_{\pi\in\mathcal{P}(l,k)} (-1)^{\pi} \prod_{\lambda=1}^{l} H_{i_{\pi(2\lambda)}-i_{\pi(2\lambda-1)}}(2t)\Phi^{(k-2l)}_{(i_{\pi(2l+1)},\ldots,i_{\pi(k)})}(t) + R_{i,L}^{(k)}(t). \tag{32}$$

Here $\mathcal{P}(l,k)$ denotes the set of permutations that correspond to choosing $0 \leqslant l \leqslant \lfloor \frac{k}{2} \rfloor$ ordered pairs from the numbers $\{1,2,\ldots,k\}$ and keeping the remaining $k-2l$ numbers in ascending order, i.e.

$$\begin{aligned}\mathcal{P}(l,k) = \{\pi \in \mathcal{S}(k) | \pi(1) < \pi(2), \pi(3) < \pi(4), \ldots, \pi(2l-1) < \pi(2l),\\ \pi(1) < \pi(3) < \cdots < \pi(2l-1), \pi(2l+1) < \pi(2l+2) < \cdots < \pi(k)\}.\end{aligned} \tag{33}$$

An expression for $H_n(t)$ is given in (C1). We also discuss relevant features of this function in appendix C. The $\Phi^{(k)}$ appearing in (32) are the homogeneous solutions (29). Finally, $R^{(k)}$ denotes the remainder term of the recursion that links to level $k-2L-2$, and is of the form

$$\begin{aligned}R_{i,L}^{(k)}(t) = \int_0^t \mathrm{d}\tau \Bigg\{ \sum_{\pi\in\mathcal{P}(L,k)} (-1)^{\pi} \prod_{\lambda=1}^{L} H_{i_{\pi(2\lambda)}-i_{\pi(2\lambda-1)}}(2(t-\tau))\\ \times \sum_{j\in N(k-2L)} \sum_{j'\in N(k-2L-2)} G^{(k-2L)}_{(i_{\pi(2L+1)},\ldots,i_{\pi(k)}),j}(t-\tau)b_{j,j'}^{(k-2L)}F_{j'}^{(k-2L-2)}(\tau)\Bigg\}.\end{aligned} \tag{34}$$

The first step in our proof that (32) is the product of iterating (31) is to consider $L=0$. The two sums in (32) collapse to a single term, that is $l=0$ and $\mathcal{P}(0,k) = \{\mathrm{Id}\}$. This obviously also applies to the first sum in (34). Defining the value of empty products in (32) and (34) as one—a convention we use throughout—then reduces (32) to (31) for $L=0$. Now assume that (32) is true for some $L \geqslant 0$. Consequently, for (32) to apply also for $L \to L+1$, the remainder has to satisfy

$$R_{i,L}^{(k)}(t) = R_{i,L+1}^{(k)}(t) + \sum_{\pi\in\mathcal{P}(L+1,k)} (-1)^{\pi} \prod_{\lambda=1}^{L+1} H_{i_{\pi(2\lambda)}-i_{\pi(2\lambda-1)}}(2t)\Phi^{(k-2(L+1))}_{(i_{\pi(2(L+1)+1)},\ldots,i_{\pi(k)})}(t). \tag{35}$$

Of course we need $L+1 \leqslant \lfloor \frac{k}{2} \rfloor - 1$ since (31) only holds on levels $k \geqslant 2$. We show below that the remainder (34) satisfies (35) which proves that (32) is the result of iterating (31).

First we evaluate the matrix product $G^{(k-2L)} \cdot b^{(k-2L)}$ in (34); we show in appendix A that this can be expressed in terms of Green's functions $G^{(k-2L-2)}$ and the functions $H_n(t)$. Substituting (A1) into (34) gives

$$R^{(k)}_{i,L}(t) = \int_0^t \mathrm{d}\tau \left\{ \sum_{\pi\in\mathcal{P}(L,k)} (-1)^{\pi} \sum_{1\leqslant\mu<\nu\leqslant k-2L} (-1)^{\nu-\mu-1} \right.$$
$$\times \left(-\frac{\partial}{\partial\tau} H_{i_{\pi(2L+\nu)}-i_{\pi(2L+\mu)}}(2(t-\tau)) \right) \prod_{\lambda=1}^{L} H_{i_{\pi(2\lambda)}-i_{\pi(2\lambda-1)}}(2(t-\tau))$$
$$\left. \times \sum_{\boldsymbol{j}\in N(k-2L-2)} G^{(k-2L-2)}_{(i_{\pi(2L+1)},\ldots,i_{\pi(k)})\setminus(i_{\pi(2L+\mu)},i_{\pi(2L+\nu)}),\boldsymbol{j}}(t-\tau) F^{(k-2L-2)}_{\boldsymbol{j}}(\tau) \right\}. \tag{36}$$

Then we express $F^{(k-2L-2)}$ in (36) via the recursion formula (31) and use the general identity for our Green's function

$$\forall\tau,\tau'>0: \sum_{\boldsymbol{j}\in N(k)} G^{(k)}_{i,\boldsymbol{j}}(\tau) G^{(k)}_{\boldsymbol{j},i'}(\tau') = G^{(k)}_{i,i'}(\tau+\tau'). \tag{37}$$

This gives

$$R^{(k)}_{i,L}(t) = \int_0^t \mathrm{d}\tau \left\{ \sum_{\pi\in\mathcal{P}(L,k)} (-1)^{\pi} \sum_{1\leqslant\mu<\nu\leqslant k-2L} (-1)^{\nu-\mu-1} \right.$$
$$\times \left(-\frac{\partial}{\partial\tau} H_{i_{\pi(2L+\nu)}-i_{\pi(2L+\mu)}}(2(t-\tau)) \right) \prod_{\lambda=1}^{L} H_{i_{\pi(2\lambda)}-i_{\pi(2\lambda-1)}}(2(t-\tau))$$
$$\times \left[\sum_{\boldsymbol{j}\in N(k-2L-2)} G^{(k-2L-2)}_{(i_{\pi(2L+1)},\ldots,i_{\pi(k)})\setminus(i_{\pi(2L+\mu)},i_{\pi(2L+\nu)}),\boldsymbol{j}}(t) A^{(k-2L-2)}_{\boldsymbol{j}} \right.$$
$$+ \int_0^\tau \mathrm{d}\tau' \sum_{\boldsymbol{j}\in N(k-2L-2)} \sum_{\boldsymbol{j}'\in N(k-2L-4)}$$
$$\left. \left. \times G^{(k-2L-2)}_{(i_{\pi(2L+1)},\ldots,i_{\pi(k)})\setminus(i_{\pi(2L+\mu)},i_{\pi(2L+\nu)}),\boldsymbol{j}}(t-\tau') b^{(k-2L-2)}_{\boldsymbol{j},\boldsymbol{j}'} F^{(k-2L-4)}_{\boldsymbol{j}'}(\tau') \right] \right\}. \tag{38}$$

In (38) τ-dependencies via the Green functions have dropped out.

Next, we rewrite the permutations in (38) in terms of those from the set $\mathcal{P}(L+1,k)$. Consider the case $L \geqslant 1$ first. According to (33) the first $2L$ values of $\pi \in \mathcal{P}(L,k)$ correspond to L ordered pairs chosen from $\{1,\ldots,k\}$ whereas $\pi(2L+1),\ldots,\pi(k)$ are the remaining numbers in ascending order. Now the μ,ν sum chooses another ordered pair $\pi(2L+\mu) < \pi(2L+\nu)$ since $\mu<\nu$ and π is ordered in that range. In contrast to the L pairs selected by π the new pair does not obey the mutual ordering between pairs. As both sums run over all possible choices we get each set of $L+1$ pairs $L+1$ times, with $\pi(2L+\mu)$ in all mutual orderings from $\pi(2L+\mu) < \pi(1) < \cdots < \pi(2L-1)$ to $\pi(1) < \cdots < \pi(2L-1) < \pi(2L+\mu)$. But this ordering is only relevant for the second line in (38). Hence we may restrict the choice of pairs to those contained in $\mathcal{P}(L+1,k)$ and confine the sum over the $L+1$ ways of selecting a 'special' pair to the second line in (38)—which thereby turns into the derivative of the product of $L+1$ terms

$$R^{(k)}_{\boldsymbol{i},L}(t) = \int_0^t \mathrm{d}\tau \left\{ \sum_{\pi\in\mathcal{P}(L+1,k)} (-1)^{\pi} \left(-\frac{\partial}{\partial\tau} \prod_{\lambda=1}^{L+1} H_{i_{\pi(2\lambda)}-i_{\pi(2\lambda-1)}}(2(t-\tau)) \right) \right.$$
$$\times \left[\sum_{\boldsymbol{j}\in N(k-2(L+1))} G^{(k-2(L+1))}_{(i_{\pi(2(L+1)+1)},\ldots,i_{\pi(k)}),\boldsymbol{j}}(t) A^{(k-2(L+1))}_{\boldsymbol{j}} \right.$$
$$+ \int_0^\tau \mathrm{d}\tau' \sum_{\boldsymbol{j}\in N(k-2(L+1))} \sum_{\boldsymbol{j}'\in N(k-2(L+1)-2)}$$
$$\left. \left. \times G^{(k-2(L+1))}_{(i_{\pi(2(L+1)+1)},\ldots,i_{\pi(k)}),\boldsymbol{j}}(t-\tau') b^{(k-2(L+1))}_{\boldsymbol{j},\boldsymbol{j}'} F^{(k-2(L+1)-2)}_{\boldsymbol{j}'}(\tau') \right] \right\}. \tag{39}$$

Now we convince ourselves that the sign of the permutation in (39) matches the sign-factor in (38): in (38) we start from the ordered vector $(1, 2, \ldots, k)$. Applying $\pi \in \mathcal{P}(L, k)$ permutes this vector as described, with the sign of the permutation being $(-1)^{\pi}$. The sum over μ, ν then chooses components $2L + \mu$, $2L + \nu$ as pair number $L + 1$. Now it takes $\nu - \mu - 1$ transpositions to move those components next to each other and an even number of further ones—thus irrelevant—to move this pair around. Altogether, $(-1)^{\pi}(-1)^{\nu-\mu-1}$ in (38) is the sign of the corresponding permutation $\pi \in \mathcal{P}(L + 1, k)$ in (39).

In the $L = 0$ case, equivalence of (38), (39) is obvious: since $\mathcal{P}(0, k) = \{\mathrm{Id}\}$ we can remove this sum and all π from (38); the empty product over λ is one by definition. So (38) is just the explicit formulation of the sum over $\pi \in \mathcal{P}(1, k)$ in (39).

Finally an integration by parts in τ turns (39) into (35), using $H_n(0) = 0$. This completes the proof that (32) is the result of iterating (31).

Having proved (32), we can now set $L = \lfloor \frac{k}{2} \rfloor - 1$. The remainder term in (32) then links to level 0 or 1 of the hierarchy for k even or odd, respectively. But, as discussed below (19), we know the functions $F^{(0)} = A^{(0)}$ and $F^{(1)} = \Phi^{(1)}$ appearing in the remainder. In fact, it can be simplified in full analogy with the above calculation for $L < \lfloor \frac{k}{2} \rfloor - 1$ and this yields the *general*, *explicit* solution of the hierarchy (17) as

$$F^{(k)}_{\boldsymbol{i}}(t) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{\pi\in\mathcal{P}(l,k)} (-1)^{\pi} \prod_{\lambda=1}^{l} H_{i_{\pi(2\lambda)}-i_{\pi(2\lambda-1)}}(2t) \Phi^{(k-2l)}_{(i_{\pi(2l+1)},\ldots,i_{\pi(k)})}(t). \tag{40}$$

The homogeneous solutions $\Phi^{(k)}$ appearing in (40) are given by (29) for $1 \leqslant k \leqslant N$. We have incorporated the link to level 0 in (40), which occurs for even k, by defining $\Phi^{(0)} = A^{(0)}$. The definition of the sets $\mathcal{P}(l, k)$ may be found in (33) and expressions for the functions $I_n(x)$—contained in the $\Phi^{(k)}$—and $H_n(t)$ are given in (B1) and (C1), respectively. According to (40) all functions $I_n(x)$ and $H_n(t)$ appearing in $F^{(k)}$ are associated with (even) odd levels for (even) odd k and consequently the appropriate sets $Q = Q_{\mathrm{e}}$ or $Q = Q_{\mathrm{o}}$ as given by equations (25), (26) must be substituted in their representations. Explicit versions of (40) for the first few levels read, bearing in mind that the product over λ equals one for $l = 0$,

$$F^{(1)}_i(t) = \Phi^{(1)}_i(t)$$
$$F^{(2)}_{\boldsymbol{i}}(t) = \Phi^{(2)}_{\boldsymbol{i}}(t) + H_{i_2-i_1}(2t)\Phi^{(0)}$$
$$F^{(3)}_{\boldsymbol{i}}(t) = \Phi^{(3)}_{\boldsymbol{i}}(t) + H_{i_2-i_1}(2t)\Phi^{(1)}_{i_3}(t) - H_{i_3-i_1}(2t)\Phi^{(1)}_{i_2}(t) + H_{i_3-i_2}(2t)\Phi^{(1)}_{i_1}(t)$$

$$F^{(4)}_{\boldsymbol{i}}(t) = \Phi^{(4)}_{\boldsymbol{i}}(t) + H_{i_2-i_1}(2t)\Phi^{(2)}_{(i_3,i_4)}(t) - H_{i_3-i_1}(2t)\Phi^{(2)}_{(i_2,i_4)}(t) + H_{i_4-i_1}(2t)\Phi^{(2)}_{(i_2,i_3)}(t)$$
$$+ H_{i_3-i_2}(2t)\Phi^{(2)}_{(i_1,i_4)}(t) - H_{i_4-i_2}(2t)\Phi^{(2)}_{(i_1,i_3)}(t) + H_{i_4-i_3}(2t)\Phi^{(2)}_{(i_1,i_2)}(t)$$
$$+ H_{i_2-i_1}(2t)H_{i_4-i_3}(2t)\Phi^{(0)} - H_{i_3-i_1}(2t)H_{i_4-i_2}(2t)\Phi^{(0)}$$
$$+ H_{i_4-i_1}(2t)H_{i_3-i_2}(2t)\Phi^{(0)}.$$

We remind the reader that the functions $F^{(k)}$ only have direct physical meaning if the indices $\boldsymbol{i}$ are chosen from $N(k)$. Extended over all indices $\boldsymbol{i} \in \{1, \ldots, N\}^k$ they are permutationally antisymmetric; this follows from (31) and the corresponding property of the Green function. Therefore the $F^{(k)}$ at any time t can be used directly as antisymmetrized initial conditions $\bar{A}^{(k)}$ in (28); this property will be useful below.

Equation (40) is evidently the simplest possible representation of $F^{(k)}$ for generic initial conditions $A^{(k)}$ as encoded in the $\Phi^{(k)}$. While for the generic case the complexity of $F^{(k)}$ grows rapidly with the level k, we will see below that in physically interesting cases significant simplifications occur.

2.3. The thermodynamic limit

The entire analysis presented in section 2 can be repeated in full analogy for an infinite spin chain. There, one takes the thermodynamic limit in (4), which thereby becomes an infinite hierarchy of differential equations. The individual levels k may again be rewritten in the matrix form (17) over ordered indices. For the infinite chain, however, one chooses $-\infty < i_1 < i_2 < \cdots < i_k < +\infty$ to obtain complete equations. The infinite matrices $a^{(k)}$ and $b^{(k)}$ have a slightly simpler structure since the shifts $\boldsymbol{i} - \boldsymbol{e}_1$ and $\boldsymbol{i} + \boldsymbol{e}_k$ can never produce an index pair. In other words N-periodicity of the finite ring is lost. Nevertheless (17) may be treated as an inhomogeneous system of differential equations which can be solved according to the steps presented in sections 2.1 and 2.2. Rather than the discrete N-(anti)-periodic Fourier transforms (23), (24) the standard Fourier series expansion is useful for solving the homogeneous equations. The structure of the solutions (40) of the hierarchy and the homogeneous solutions (29) remains completely unaffected. But instead of the functions $I_n(x)$, $H_n(t)$ their $N \to \infty$ limits $\mathrm{I}_n(x)$, $\mathrm{H}_n(t)$ discussed in appendix B, C appear in (29), (40). The sum over the initial conditions in (29) obviously runs over $-\infty < j_1 < \cdots < j_k < \infty$ for an infinite chain.

Alternatively, the rigorous solutions (29), (40) for the finite ring also produce the same $N \to \infty$ limit. This is evident for (40) as it comprises a finite number of functions, the limit of each of which exists. In (29), however, one has to keep in mind that the functions $I_n(x)$ are N-(anti)-periodic. Therefore all summation indices $\boldsymbol{j}$ that are close to $\boldsymbol{i}$ *modulo* N contribute to the sum in (29); the site N for instance is a neighbour of site 1. So a meaningful $N \to \infty$ limit can only be taken if we first shift the index range of the finite ring. An appropriate replacement for $N(k)$ is for instance $-\lfloor \frac{N-1}{2} \rfloor \leqslant i_1 < \cdots < i_k \leqslant \lfloor \frac{N}{2} \rfloor$. Shifting the index range on the ring changes the domains of the solutions (29), (40) but leaves them unaffected otherwise. Convergence for $N \to \infty$ of the sum in (29) over the symmetric index range can then be proved [16], for any fixed index vector $\boldsymbol{i}$.

In the following we continue to analyse the finite ring of spins. But according to the above discussion the thermodynamic limit may be taken at any stage. It just amounts to replacing the functions $I_n(x)$, $H_n(t)$ by their $N \to \infty$ limits $\mathrm{I}_n(x)$, $\mathrm{H}_n(t)$ and extending summation ranges. From now on we omit the limits for summations that are taken over the ring, as for instance in (28). In a finite system these summations run over $1, \ldots, N$ while in the thermodynamic limit they are to be evaluated over $-\infty, \ldots, +\infty$.

3. Equilibrium correlations

As discussed in the beginning of section 2 the functions $F^{(k)}(t) = C^{(k)}(t)$ describe the evolution of correlations if we use the initial conditions $A^{(k)} = C^{(k)}(0)$. Here we show that our solution (40) reproduces the well-known equilibrium correlation functions for $T > 0$.

Equilibrium correlations are obtained from the dynamical solutions (40) by taking the limit $t \to \infty$. Let us first focus on the homogeneous solutions $\Phi^{(k)}$. It is convenient to consider the representation (28) instead of (29); recall that for $\boldsymbol{i} \in N(k)$ we have $\bar{\Phi}^{(k)} = \Phi^{(k)}$. When taking the modulus of (28) and using the triangular inequality we may drop the initial conditions $|\bar{A}^{(k)}| \leqslant |C^{(k)}(0)| \leqslant 1$. This gives the first inequality in

$$\left|\Phi_{\boldsymbol{i}}^{(k)}(t)\right| \leqslant \sum_{j_1,\ldots,j_k} \prod_{\eta=1}^{k} \mathrm{e}^{-t} \left|I_{i_\eta - j_\eta}(\gamma t)\right| = \prod_{\eta=1}^{k} \mathrm{e}^{-t} \sum_{j_\eta} \left|I_{i_\eta - j_\eta}(\gamma t)\right| \leqslant \mathrm{e}^{-kt/\tau_{\mathrm{eq}}}. \tag{41}$$

As indicated, the k-dimensional sum in (41) factorizes once the initial condition terms are removed. Then, using the bound (B10) on $\sum_n |I_n(x)|$ gives the second inequality in (41) for any system size N. Here $1/\tau_{\mathrm{eq}} = 1 - \gamma$ is the smallest eigenvalue of the master operator [13] corresponding to (1). Since $T > 0$ implies $\gamma = \tanh(2J/T) < 1$ the equilibration time τ_{eq} is finite. Therefore at any level $k \geqslant 1$ of the hierarchy the homogeneous solutions vanish in equilibrium. At level $k = 0$, however, we have $\Phi^{(0)}(t) = A^{(0)} = C^{(0)} = 1$ at all times.

Next consider the $t \to \infty$ limit of the functions $H_n(t)$. Based on the representation (C2) it is straightforward to show that for $T > 0$ and arbitrary system size N we are left with the sum

$$H_{n,\mathrm{eq}} = \lim_{t\to\infty} H_n(t) = \frac{1}{N} \sum_{q\in Q} \sin(nq) \frac{\gamma \sin q}{1 - \gamma \cos q}. \tag{42}$$

By combining the results (41) and (42) we now obtain the equilibrium correlations from (40). Any correlation function $C^{(2k+1)}$ containing an odd number of spins is expressed in terms of $\Phi^{(1)}, \Phi^{(3)}, \ldots, \Phi^{(2k+1)}$. But all these homogeneous solutions vanish in equilibrium and hence so do the correlations of odd order, as expected. For equilibrium correlations of even order $C^{(2k)}$, on the other hand, the only nonzero terms in (40) are those having a factor $\Phi^{(0)} = 1$. This gives

$$C_{\boldsymbol{i},\mathrm{eq}}^{(2k)} = \sum_{\pi\in\mathcal{P}(k)} (-1)^{\pi} \prod_{\lambda=1}^{k} H_{i_{\pi(2\lambda)} - i_{\pi(2\lambda-1)},\mathrm{eq}} \tag{43}$$

where we have introduced the notation $\mathcal{P}(k) = \mathcal{P}(k, 2k)$ for the usual set of ordered pairings. Since the functions $H_{n,\mathrm{eq}}$ in (43) are associated with even levels we need to set $Q = Q_{\mathrm{e}}$ when substituting (42). For ordered indices $\boldsymbol{i} \in N(2k)$ the result (43) describes multispin equilibrium correlations in the finite Glauber–Ising model at $T > 0$. Over non-ordered indices $\boldsymbol{i} \in \{1, \ldots, N\}^{2k}$, however, (43) carries the permutational antisymmetry of (40); this property will be useful below.

The link between (43) and the result obtained from a transfer-matrix calculation is established by realizing that for $1 \leqslant n \leqslant N - 1$ we have for (42) the identity

$$\frac{1}{N} \sum_{q\in Q_{\mathrm{e}}} \sin(nq) \frac{\gamma \sin q}{1 - \gamma \cos q} = \frac{[\tanh(J/T)]^n + [\tanh(J/T)]^{N-n}}{1 + [\tanh(J/T)]^N}. \tag{44}$$

The lhs of (44) is identically zero for $n = 0$, N and N-antiperiodic, however. To prove (44) one notes that the sum may be rewritten in the form of the inverse Fourier transform (24). So we may equivalently prove the converse statement (23) for $q \in Q_{\mathrm{e}}$. The latter just involves a

geometric summation and is straightforward. We finally remark that when the representation (44) is used for $H_{n,\mathrm{eq}}$, the permutational antisymmetry of (43) is lost. Only for ordered index vectors $\boldsymbol{i} \in N(2k)$ are all combinations $n = i_{\pi(2\lambda)} - i_{\pi(2\lambda-1)}$ in (43) in the range where (44) applies.

4. Correlations after a quench

Next we derive dynamical correlation functions $C^{(k)}(t)$ from our general solution (40). We consider the example where the system is quenched at $t = 0$ from an equilibrium state at $\gamma_\mathrm{i} < 1$, corresponding to $T_\mathrm{i} > 0$, to some arbitrary temperature γ. As in section 3 we may identify $F^{(k)}(t) = C^{(k)}(t)$ if we use equilibrium correlations as the initial conditions $A^{(k)}$.

On odd levels of the hierarchy we have $A^{(2k+1)} = 0$ since equilibrium correlations between any odd number of spins vanish. Consequently $\Phi^{(2k+1)} = 0$ according to (29) and via (40) also $C^{(2k+1)}(t) = 0$ at all times. This conclusion is trivial because the master equation (1) and the equilibrium initial state are invariant under spin inversion $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$. On even levels $2k \geqslant 2$ of the hierarchy we express $A^{(2k)}$ via (42) and (43) as

$$\bar{A}^{(2k)}_{\boldsymbol{i}} = \sum_{\pi\in\mathcal{P}(k)} (-1)^\pi \prod_{\lambda=1}^{k} \frac{1}{N} \sum_{q\in Q_\mathrm{e}} \sin[q(i_{\pi(2\lambda)} - i_{\pi(2\lambda-1)})] \frac{\gamma_\mathrm{i} \sin q}{1 - \gamma_\mathrm{i} \cos q}. \tag{45}$$

As pointed out below (43), this representation for the equilibrium correlation functions is permutationally antisymmetric and consequently zero for non-pairwise distinct indices $\boldsymbol{i}$. This is why we have been able to write $\bar{A}$ instead of A in (45). Knowing $\bar{A}$ is convenient as it allows us to derive the homogeneous solutions from (28) instead of the more complex expression (29). In fact, since (45) is factorized over index pairs, the sum in (28) also factorizes into two-dimensional sums

$$\Phi^{(2k)}_{\boldsymbol{i}}(t) = \sum_{\pi\in\mathcal{P}(k)} (-1)^\pi \prod_{\lambda=1}^{k} \left[\sum_{m,n} \mathrm{e}^{-2t} I_{i_{\pi(2\lambda)}-m}(\gamma t) I_{i_{\pi(2\lambda-1)}-n}(\gamma t) \frac{1}{N} \right.$$
$$\left. \times \sum_{q\in Q_\mathrm{e}} \sin[q(m-n)] \frac{\gamma_\mathrm{i} \sin q}{1 - \gamma_\mathrm{i} \cos q} \right]. \tag{46}$$

Using the properties (B7), (B8) of the functions $I_n(x)$, the sum $\sum_{m,n}$ in (46) can easily be evaluated. Here it is essential that the functions $I_n(x)$ in (46) are associated with the even levels $2k$ and that in the argument of $\sin[q(m-n)]$ we have $q \in Q_\mathrm{e}$ too, as (B7) would not apply otherwise. We obtain

$$\Phi^{(2k)}_{\boldsymbol{i}}(t) = \sum_{\pi\in\mathcal{P}(k)} (-1)^\pi \prod_{\lambda=1}^{k} H'_{i_{\pi(2\lambda)}-i_{\pi(2\lambda-1)}}(2t) \tag{47}$$

where $H'_n(t)$ is the result of the summation over m, n and may be written as

$$H'_n(t) = \frac{1}{N} \sum_{q\in Q_\mathrm{e}} \sin(nq) \frac{\gamma_\mathrm{i} \sin q}{1 - \gamma_\mathrm{i} \cos q} \mathrm{e}^{-t(1-\gamma\cos q)}. \tag{48}$$

We remark that this straightforward derivation of the homogeneous solutions heavily rests upon the permutational antisymmetry of our expression (43) for equilibrium correlations. When Glauber derived the dynamics of two-spin correlations [1] he expressed the equilibrium values via (44). He introduced images at this stage of the calculation to make (44) permutationally antisymmetric.

Having worked out the homogeneous solutions (47) that correspond to our problem, we now obtain the correlations after the quench from (40)

$$C_{\boldsymbol{i}}^{(2k)}(t) = \sum_{l=0}^{k} \sum_{\pi\in\mathcal{P}(l,2k)} (-1)^{\pi} \sum_{\hat{\pi}\in\mathcal{P}(k-l)} (-1)^{\hat{\pi}} \prod_{\lambda=1}^{l} H_{i_{\pi(2\lambda)}-i_{\pi(2\lambda-1)}}(2t) \times \prod_{\lambda'=1}^{k-l} H'_{i_{\pi(2l+\hat{\pi}(2\lambda'))}-i_{\pi(2l+\hat{\pi}(2\lambda'-1))}}(2t). \tag{49}$$

For the term with $l = k$ to correctly give the contribution from $\Phi^{(k-l)} = C^{(0)} = 1$ one has to define $\mathcal{P}(0) = \{\mathrm{Id}\}$ here. To simplify (49), we now focus on the combination of pairings that occur for some fixed l: each $\pi \in \mathcal{P}(l, 2k)$ draws l ordered pairs from $\{1, \ldots, 2k\}$. These define the indices of the factors $H_n(2t)$ in (49). Out of the remaining, ordered $2k-2l$ numbers $\pi(2l+1), \ldots, \pi(2k)$, the permutation $\hat{\pi} \in P(k-l)$ selects another $k-l$ ordered pairs, which give the indices of the factors $H'_n(2t)$. If we denote by $\pi_l \in \mathcal{S}(2k)$ the overall permutation on $\{1, \ldots, 2k\}$, then all π_l satisfy

$$\begin{aligned} &\pi_l(1) < \pi_l(2), \pi_l(3) < \pi_l(4), \ldots, \pi_l(2k-1) < \pi_l(2k) \\ &\pi_l(1) < \pi_l(3) < \ldots < \pi_l(2l-1), \pi_l(2l+1) < \pi_l(2l+3) < \ldots < \pi_l(2k-1). \end{aligned} \tag{50}$$

By ordering the first l pairs against the subsequent $k-l$ ones, each $\pi_l \in \mathcal{S}(2k)$ can be mapped onto an ordered pairing $\bar{\pi} \in \mathcal{P}(k)$. This leaves the sign of the permutation unaffected as it takes an even number of transpositions to reorder pairs, i.e. $(-1)^{\bar{\pi}} = (-1)^{\pi_l} = (-1)^{\pi}(-1)^{\hat{\pi}}$. Conversely, we have that for each $\bar{\pi} \in \mathcal{P}(k)$ there are $\binom{k}{l}$ semi-ordered pairings π_l of the form (50). These correspond to the different ways of distributing $k-l$ primes among the functions $H_n(2t)$ in (49) when we choose the index pairs according to $\bar{\pi}$. Together with the summation over l we can thus factorize (49) for each $\bar{\pi}$, giving

$$C_{\boldsymbol{i}}^{(2k)}(t) = \sum_{\bar{\pi}\in\mathcal{P}(k)} (-1)^{\bar{\pi}} \prod_{\lambda=1}^{k} H''_{i_{\bar{\pi}(2\lambda)}-i_{\bar{\pi}(2\lambda-1)}}(2t) \tag{51}$$

where $H''_n(t) = H_n(t) + H'_n(t)$. Since the functions $H_n(t)$ in (49) are associated with even levels $2k$ we need to set $Q = Q_{\mathrm{e}}$ in the representation (C2). Together with (48) this yields

$$H''_n(t) = \frac{1}{N} \sum_{q\in Q_{\mathrm{e}}} \sin(nq) \left\{ \frac{\gamma_{\mathrm{i}} \sin q}{1-\gamma_{\mathrm{i}} \cos q} \mathrm{e}^{-t(1-\gamma\cos q)} + \frac{\gamma \sin q}{1-\gamma\cos q} [1 - \mathrm{e}^{-t(1-\gamma\cos q)}] \right\}. \tag{52}$$

At $t = 0$ our solution for dynamical correlations (51), (52) obviously reproduces the equilibrium values (42), (43) for the initial temperature γ_{i}. Note that for a quench from $T_{\mathrm{i}} = \infty$ ($\gamma_{\mathrm{i}} = 0$), i.e. a random initial configuration, (52) reduces to $H''_n(t) = H_n(t)$. It is equally clear that we recover the equilibrium values at γ in the limit $t \to \infty$. The spectrum of relaxation times for correlations is encoded in the exponents in (52) and depends on the final temperature γ after the quench and the system size via Q_{e} as given in (25). In the thermodynamic limit $N \to \infty$ we can replace $\frac{1}{N}\sum_q$ by $\frac{1}{2\pi}\int \mathrm{d}q$ in (52) and, for consistency, then use the symbol $\mathrm{H}''_n(t)$. Further properties of $H''_n(t)$ are discussed in appendix C. An equivalent formulation of the results (51), (52) for the infinite spin chain was derived in [2].

It is worthwhile noting that the general solution (40) preserves the structure of ordered pairings. In the case at hand, for instance, we used the initial conditions (43) and arrived at the result (51). Both expressions have the same structure and (51) again carries the permutational antisymmetry of (40). Therefore, the dynamical correlations after an arbitrary sequence of temperature shifts are still of the form (51) and could be written immediately as long as the

system is initially in an equilibrium state. Examples of two and four-spin correlation functions as obtained from (51) read

$$C^{(2)}_{\boldsymbol{i}}(t) = H''_{i_2-i_1}(2t)$$
$$C^{(4)}_{\boldsymbol{i}}(t) = H''_{i_2-i_1}(2t)H''_{i_4-i_3}(2t) - H''_{i_3-i_1}(2t)H''_{i_4-i_2}(2t) + H''_{i_4-i_1}(2t)H''_{i_3-i_2}(2t).$$

The number of terms grows as the number of permutations contained in $\mathcal{P}(k)$, i.e., $(2k-1)!! = 1\cdot 3\cdots(2k-1)$, giving 15 terms for $C^{(6)}$, 105 terms for $C^{(8)}$ and so on.

5. Two-time correlations after a quench

As discussed above, the evolution equations (8) of two-time correlations $C^{(k,l)}(t,t')$ may be written in the form (17). Hence these functions again follow from the general solution (40) if we use the equal time initial conditions (9) at $t = t'$; obviously time arguments t in (40) have to be replaced by $\Delta = t - t'$ in this setup. We consider the same quench as in section 4 because this allows us to express the initial conditions (9) in terms of (51) for even $k + l$. On all levels k where $k + l$ is odd, that is all even levels k for odd l and vice versa, we have $C^{(k,l)}(t',t') = C^{(k+l)}(t') = 0$. Hence $C^{(k,l)}(t,t') = 0$ at all times from (29), (40); this is again just a consequence of the spin inversion symmetry $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$. So from now on we assume that $k + l$ is even.

In analogy with section 4 an expression $\bar{A}$ for the equal time initial conditions (9) that is permutationally antisymmetric in $\boldsymbol{i}$ would be desirable. Without loss of generality we require that the vector $\boldsymbol{j}$—only appearing as a parameter—is ordered, $\boldsymbol{j} \in N(l)$. Now the following problems have to be handled. First, if $\boldsymbol{i} \in N(k)$ is also ordered, our expression for $\bar{A}$ has to produce the correlation function between the $k + l$ sites $\boldsymbol{i} \cup \boldsymbol{j}$. But the combined vector is not necessarily ordered and hence the associated correlation cannot be expressed directly via (51). Second, there could still be pairs of equal components between $\boldsymbol{i}$ and $\boldsymbol{j}$. While the correlation over $\boldsymbol{i} \cup \boldsymbol{j}$ then reduces its order via the hierarchical property (5), the expression (51) is identically zero. A systematic construction of $\bar{A}$ based on (5) and (51) is possible but rather cumbersome. Let us therefore immediately state the result for $\bar{A}$ and prove its correctness instead:

$$\bar{A}^{(k,l)}_{\boldsymbol{i},\boldsymbol{j}}(t') = \sum_{\pi\in\mathcal{P}(\frac{k+l}{2})} (-1)^{\pi} \prod_{\lambda=1}^{\frac{k+l}{2}} \mathcal{U}^{\boldsymbol{j}}_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)}}(t') \tag{53}$$

where

$$\mathcal{U}^{\boldsymbol{j}}_{a,b}(t') = \begin{cases} H''_{j_\nu - j_\mu}(2t') & (a,b) = (j_\mu, j_\nu) \\ (-1)^{\nu-1}\delta_{i_\varepsilon,j_\nu} + \left[\prod_{\lambda=1}^{l} \operatorname{sgn}(j_\lambda - i_\varepsilon)\right] H''_{j_\nu - i_\varepsilon}(2t') & \text{for} \quad (a,b) = (i_\varepsilon, j_\nu) \\ \left[\prod_{\lambda=1}^{l} \operatorname{sgn}(j_\lambda - i_\varepsilon)\operatorname{sgn}(j_\lambda - i_\delta)\right] H''_{i_\delta - i_\varepsilon}(2t') & (a,b) = (i_\varepsilon, i_\delta). \end{cases} \tag{54}$$

The $\operatorname{sgn}(n)$ in (54) are the standard sign functions, with $\operatorname{sgn}(n) = -\operatorname{sgn}(-n) = 1$ for $n > 0$ and $\operatorname{sgn}(0) = 0$. To illustrate the notation of (53) and (54) we consider as a simple example the 6-spin two-time correlation with $k = l = 3$. In this case the sum in (53) runs over the 15 ordered pairings contained in $\mathcal{P}(3)$. Among these focus on the particular pairing π that gives $(1,3,2,4,5,6)$. Here the indices (a,b) of the 3 factors $\mathcal{U}$ in (53) are $((\boldsymbol{i}\cup\boldsymbol{j})_1, (\boldsymbol{i}\cup\boldsymbol{j})_3) = (i_1, i_3)$, $((\boldsymbol{i}\cup\boldsymbol{j})_2, (\boldsymbol{i}\cup\boldsymbol{j})_4) = (i_2, j_1)$ and $((\boldsymbol{i}\cup\boldsymbol{j})_5, (\boldsymbol{i}\cup\boldsymbol{j})_6) = (j_2, j_3)$. According to (54) we substitute $\prod_\lambda \operatorname{sgn}(j_\lambda - i_1)\operatorname{sgn}(j_\lambda - i_3) H''_{i_3-i_1}(2t')$ for the first factor $\mathcal{U}_{i_1,i_3}$

as both indices are drawn from $\boldsymbol{i}$. The second factor $\mathcal{U}_{i_2,j_1}$ comprises components from $\boldsymbol{i}$ as well as $\boldsymbol{j}$ and must therefore be expressed as $(-1)^{1-1}\delta_{i_2,j_1} + \prod_\lambda \mathrm{sgn}(j_\lambda - i_2) H''_{j_1-i_2}(2t')$. The third factor, with both indices from $\boldsymbol{j}$, is simply $\mathcal{U}_{j_2,j_3} = H''_{j_3-j_2}(2t')$. The remaining terms in the sum over $\pi \in \mathcal{P}(3)$ in (53) are worked out similarly.

A few further comments on (54) are in order. First, the case $(a, b) = (j_\nu, i_\varepsilon)$ cannot occur in (54) because all pairs in π are by definition ordered. Second, the $\mathcal{U}$ carry the superscript $\boldsymbol{j}$ since the products in (54) run over the full vector $\boldsymbol{j}$, so that $\mathcal{U}$ depends on all components of $\boldsymbol{j}$ except in the first case of (54). Finally, the factor $(-1)^{\nu-1}$ in (54) depends on the actual number ν of the component j_ν drawn from $\boldsymbol{j}$.

Having clarified the interpretation of (53) and (54) we now prove that these equations are indeed the initial conditions $\bar{A}$ corresponding to (9). Let us first show that our expressions are antisymmetric under permutations in $\boldsymbol{i} \in \{1, \ldots, N\}^k$. It is sufficient to prove antisymmetry under transposition of any two adjacent components of $\boldsymbol{i}$, say $i_\mu, i_{\mu+1}$, since all permutations may be decomposed into such transpositions. Now, given $i_\mu, i_{\mu+1}$, we split the pairings $\mathcal{P}$ in (53) into those $\mathcal{P}'$ containing the pair $(i_\mu, i_{\mu+1})$ and the remaining pairings $\mathcal{P}''$ where i_μ and $i_{\mu+1}$ are distributed over separate pairs. For each $\pi \in \mathcal{P}'$ there is a factor $\mathcal{U}_{i_\mu,i_{\mu+1}}$ in (53) which, according to (54), is of the form $\prod_\lambda \mathrm{sgn}(j_\lambda - i_\mu)\mathrm{sgn}(j_\lambda - i_{\mu+1}) H''_{i_{\mu+1}-i_\mu}(2t')$. With $H''_{-n}(t) = -H''_n(t)$ this immediately shows that $\mathcal{U}_{i_{\mu+1},i_\mu} = -\mathcal{U}_{i_\mu,i_{\mu+1}}$; therefore each term in (53) with $\pi \in \mathcal{P}'$ is antisymmetric under $i_\mu \leftrightarrow i_{\mu+1}$. For $\pi \in \mathcal{P}''$ we have to apply a different reasoning. Suppose, for instance, that a particular pairing $\pi \in \mathcal{P}''$ contains the pairs (i_ε, i_μ) and $(i_\delta, i_{\mu+1})$. As the pairs are ordered we have $\varepsilon < \mu, \delta < \mu + 1$ and the mutual ordering $\varepsilon < \delta$. But $\delta \neq \mu$ implies $\delta < \mu$ and hence the conditions $\varepsilon < \mu + 1, \delta < \mu$ and $\varepsilon < \delta$ also hold. So there is another pairing $\pi' \in \mathcal{P}''$ that contains $(i_\varepsilon, i_{\mu+1})$ and (i_δ, i_μ) but is identical to π otherwise. Since π and π' have opposite signs and because the transposition $i_\mu \leftrightarrow i_{\mu+1}$ maps the pairings corresponding to π and π' onto each other, the sum of the two associated terms in (53) is antisymmetric under $i_\mu \leftrightarrow i_{\mu+1}$. This argument is readily extended to all $\pi \in \mathcal{P}''$ by considering the remaining cases; these are pairs of the form $(i_\varepsilon, i_\mu), (i_{\mu+1}, x)$ where $\varepsilon < \mu$ and $x \in \{i_{\mu+2}, \ldots, i_k, j_1, \ldots, j_l\}$ or $(i_\mu, x), (i_{\mu+1}, y)$ with $x, y \in \{i_{\mu+2}, \ldots, j_l\}$.

Now we turn to ordered index vectors $\boldsymbol{i} \in N(k)$. Then (53) must produce the correlation function over the sites $\boldsymbol{i} \cup \boldsymbol{j}$ according to the initial condition (9). Let us for the moment focus on those $\boldsymbol{i} \in N(k), \boldsymbol{j} \in N(l)$ for which the combined vector $\boldsymbol{i} \cup \boldsymbol{j}$ has pairwise distinct components. This allows us to ignore the Kronecker delta in (54). Thus $\mathcal{U}^{\boldsymbol{j}}_{a,b}$ contains a factor $\prod_\lambda \mathrm{sgn}(j_\lambda - i_\varepsilon)$ whenever the index a is drawn from $\boldsymbol{i}$, i.e. when $a = i_\varepsilon$, and similarly for b. Since all components of $\boldsymbol{i}$ occur exactly once in the product over λ in (53) we can therefore rewrite (53) as

$$\bar{A}^{(k,l)}_{\boldsymbol{i},\boldsymbol{j}}(t') = \prod_{\mu=1}^{k}\prod_{\nu=1}^{l} \mathrm{sgn}(j_\nu - i_\mu) \sum_{\pi \in \mathcal{P}(\frac{k+l}{2})} (-1)^\pi \prod_{\lambda=1}^{\frac{k+l}{2}} H''_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)} - (\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)}}(2t'). \tag{55}$$

Apart from the products over the sgn's this corresponds to our result (51) for correlation functions. But (51) is antisymmetric under index permutation. Hence, if $\bar{\pi} \in \mathcal{S}(k+l)$ is the permutation that orders $\boldsymbol{i} \cup \boldsymbol{j}$, i.e. $\boldsymbol{i} \cup \boldsymbol{j} \in N_{\bar{\pi}}(k+l)$, then the pairing sum in (55) must give $(-1)^{\bar{\pi}}$ times the correlation over $\boldsymbol{i}\cup\boldsymbol{j}$. Thus the products over the sgn must coincide with $(-1)^{\bar{\pi}}$ in order to cancel the sign of $\bar{\pi}$. We can convince ourselves that this is indeed the case. Keep μ fixed and evaluate the product over ν. For each ν with $j_\nu < i_\mu$ we have $\mathrm{sgn}(j_\nu - i_\mu) = -1$ while $i_\mu < j_\nu$ gives $\mathrm{sgn}(j_\nu - i_\mu) = +1$ and is irrelevant for the product. Altogether this gives $(-1)^{n_\mu}$ where $j_{n_\mu} < i_\mu < j_{n_\mu+1}$. Since both $\boldsymbol{i} \in N(k)$ and $\boldsymbol{j} \in N(l)$ are ordered we have $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$. Now if we start ordering $\boldsymbol{i} \cup \boldsymbol{j}$ beginning with i_k, it takes n_k transpositions to turn $\boldsymbol{i} \cup \boldsymbol{j}$ into $(i_1, \ldots, i_{k-1}, j_1, \ldots, j_{n_k}, i_k, j_{n_k+1}, \ldots, j_l)$, a further n_{k-1} transpositions

to turn this into $(i_1, \ldots, i_{k-2}, j_1, \ldots, j_{n_{k-1}}, i_{k-1}, j_{n_{k-1}+1}, \ldots, j_{n_k}, i_k, j_{n_k+1}, \ldots, j_l)$ and so on. Therefore the products over the sgn give (-1) to the power of the number of transpositions necessary to order $\boldsymbol{i} \cup \boldsymbol{j}$, which is equal to $(-1)^{\hat{\pi}}$.

Finally we lift the restriction that $\boldsymbol{i} \cup \boldsymbol{j}$ must have pairwise distinct components; $\boldsymbol{i} \in N(k)$ and $\boldsymbol{j} \in N(l)$ is of course still required. Assume that there are p pairs $i_{\varepsilon_1} = j_{\delta_1}, i_{\varepsilon_2} = j_{\delta_2}, \ldots, i_{\varepsilon_p} = j_{\delta_p}$ between $\boldsymbol{i}$ and $\boldsymbol{j}$. We write $k' = k - p, l' = l - p, \boldsymbol{i}' = \boldsymbol{i} \setminus (i_{\varepsilon_1}, \ldots, i_{\varepsilon_p})$ and $\boldsymbol{j}' = \boldsymbol{j} \setminus (j_{\delta_1}, \ldots, j_{\delta_p})$. The combined vector $\boldsymbol{i}' \cup \boldsymbol{j}'$ then contains no further component pairs. According to the hierarchical property (5) equations (53) and (54) must then reduce to (55) with $k, l, \boldsymbol{i}, \boldsymbol{j}$ replaced by $k', l', \boldsymbol{i}', \boldsymbol{j}'$, respectively. The first step towards showing that this is true consists in splitting the pairings in (53) again. Now, however, we denote by $\mathcal{P}'$ the pairings that contain all the pairs $(i_{\varepsilon_q}, j_{\delta_q})$ with $q = 1, \ldots, p$ and by $\mathcal{P}''$ the remaining pairings. Let us focus on $\pi \in \mathcal{P}''$ first. For each such π at least one pair, say $(i_{\varepsilon_q}, j_{\delta_q})$, is not formed. Instead i_{ε_q} may be paired up with (i_u, i_{ε_q}), (i_{ε_q}, i_v) or (i_{ε_q}, j_w) where $w \neq \delta_q$. In either case there is a factor $\mathcal{U} = 0$ in the product in (53) since $\mathcal{U}_{i_u, i_{\varepsilon_q}}, \mathcal{U}_{i_{\varepsilon_q}, i_v}$ and $\mathcal{U}_{i_{\varepsilon_q}, j_w}$ all contain a factor $\operatorname{sgn}(j_{\delta_q} - i_{\varepsilon_q}) = 0$ according to (54); for $\mathcal{U}_{i_{\varepsilon_q}, j_w}$ the Kronecker delta is also zero since $\boldsymbol{j} \in N(l)$ is ordered and hence $j_w \neq j_{\delta_q} = i_{\varepsilon_q}$ for $w \neq \delta_q$. So each term in the sum over π in (53) is zero for $\pi \in \mathcal{P}''$. Next we turn to the pairings $\pi \in \mathcal{P}'$. Each such π may be constructed as follows. Starting from $\boldsymbol{i} \cup \boldsymbol{j} = (i_1, \ldots i_{\varepsilon_1} \ldots i_{\varepsilon_2} \ldots i_{\varepsilon_p} \ldots i_k, j_1 \ldots j_{\delta_1} \ldots j_{\delta_2} \ldots j_{\delta_p} \ldots j_l)$, we first group all pairs $(i_{\varepsilon_q}, j_{\delta_q})$ with $q = 1, \ldots, p$ into the lowest components of the combined vector. It takes $k - \varepsilon_1 + \delta_1 - 1$ transpositions to move j_{δ_1} next to i_{ε_1} and an even (irrelevant) number of further transpositions to get $(i_{\varepsilon_1}, j_{\delta_1}, i_1, \ldots i_{\varepsilon_2} \ldots i_{\varepsilon_3} \ldots i_{\varepsilon_p} \ldots i_k, j_1 \ldots j_{\delta_2} \ldots j_{\delta_3} \ldots j_{\delta_p} \ldots j_l)$. Moving j_{δ_2} next to i_{ε_2} then only takes $k - \varepsilon_2 + \delta_2 - 2$ transpositions because component j_{δ_1} is no longer contained between i_{ε_2} and j_{δ_2} in the latter vector. Clearly, regrouping $\boldsymbol{i} \cup \boldsymbol{j}$ into the form $\boldsymbol{i}_{\mathrm{rg}} = (i_{\varepsilon_1}, j_{\delta_1}, i_{\varepsilon_2}, j_{\delta_2}, \ldots, i_{\varepsilon_p}, j_{\delta_p}) \cup \boldsymbol{i}' \cup \boldsymbol{j}'$ then requires a permutation π_{rg} of sign

$$(-1)^{\pi_{\mathrm{rg}}} = \prod_{q=1}^{p} (-1)^{k - \varepsilon_q + \delta_q - q}. \tag{56}$$

Now by arranging the top $k' + l'$ components $\boldsymbol{i}' \cup \boldsymbol{j}'$ of $\boldsymbol{i}_{\mathrm{rg}}$ according to $\hat{\pi} \in \mathcal{P}((k' + l')/2)$ and ordering the pairs $(i_{\varepsilon_q}, j_{\delta_q})$ against $((\boldsymbol{i}' \cup \boldsymbol{j}')_{\hat{\pi}(2\lambda-1)}, (\boldsymbol{i}' \cup \boldsymbol{j}')_{\hat{\pi}(2\lambda)})$, any pairing $\pi \in \mathcal{P}'((k + l)/2)$ of $\boldsymbol{i} \cup \boldsymbol{j}$ may be obtained. Also, since rearranging pairs takes an even number of transpositions, $(-1)^{\pi} = (-1)^{\pi_{\mathrm{rg}}} (-1)^{\hat{\pi}}$. Using this representation for the pairings $\pi \in \mathcal{P}'$ we may now rewrite (53). First we substitute the factorization for $(-1)^{\pi}$ just given. Second, since all pairings π contain the pairs $(i_{\varepsilon_q}, j_{\delta_q})$ with $q = 1, \ldots, p$ each product in (53) has the factors $\mathcal{U}_{i_{\varepsilon_q}, j_{\delta_q}}$ which just give $(-1)^{\delta_q - 1}$ according to (54). The remaining factors are over the pairings $\hat{\pi}$ of $\boldsymbol{i}' \cup \boldsymbol{j}'$. Since $\boldsymbol{i}' \cup \boldsymbol{j}'$ has pairwise distinct components we may apply a similar reasoning as in (55) and thus obtain

$$\begin{aligned} \bar{A}^{(k,l)}_{\boldsymbol{i},\boldsymbol{j}}(t') = (-1)^{\pi_{\mathrm{rg}}} \prod_{q=1}^{p} (-1)^{\delta_q - 1} \prod_{\substack{\mu=1 \\ \mu \neq \varepsilon_1, \ldots, \varepsilon_p}}^{k} \prod_{\nu=1}^{l} \operatorname{sgn}(j_\nu - i_\mu) \\ \times \sum_{\pi \in \mathcal{P}(\frac{k'+l'}{2})} (-1)^{\pi} \prod_{\lambda=1}^{\frac{k'+l'}{2}} H''_{(\boldsymbol{i}' \cup \boldsymbol{j}')_{\pi(2\lambda)} - (\boldsymbol{i}' \cup \boldsymbol{j}')_{\pi(2\lambda-1)}}(2t'). \end{aligned} \tag{57}$$

From (56) we see that equation (57) reduces to (55) and therefore implements the hierarchical property (5) for pairs between $\boldsymbol{i}$ and $\boldsymbol{j}$ if the identity

$$1 = \prod_{q=1}^{p} \left[(-1)^{k-\varepsilon_q-q-1} \prod_{\substack{\mu=1 \\ \mu\neq\varepsilon_1,\ldots,\varepsilon_p}}^{k} \operatorname{sgn}\left(j_{\delta_q} - i_\mu\right) \right] \tag{58}$$

holds. The product over μ is easily evaluated when using that $\boldsymbol{i} \in N(k)$ and $\boldsymbol{j} \in N(l)$ are ordered and $i_{\varepsilon_q} = j_{\delta_q}$. Hence $\operatorname{sgn}\left(j_{\delta_q} - i_\mu\right) = -1$ for $\mu = \varepsilon_q + 1, \ldots, k$. Since $\mu = \varepsilon_{q+1}, \ldots, \varepsilon_p$ are omitted, the product over μ gives $(-1)^{k-\varepsilon_q-p+q}$. So the factors in the product over q become $(-1)^{-1-p}$, yielding the overall product $(-1)^{-p(p+1)} = 1$ since $p(p+1)$ is even for any integer p. This completes our proof that equations (53) and (54) are the initial conditions $\bar{A}$ corresponding to (9).

Based on expression (53) for $\bar{A}$ it is straightforward to write the corresponding homogeneous solutions Φ via (28). Every component of $\boldsymbol{i}$ in (53)—which becomes a summation variable in (28)—occurs in exactly one of the factors of the product over λ in (53), paired up with (depending on the particular pairing π) either a different component of $\boldsymbol{i}$ or with a component of the fixed vector $\boldsymbol{j}$. Once the k-dimensional summation over $\boldsymbol{i}$ from (28) is exchanged with the summation over π from (53), it thus factorizes into one- and two-dimensional sums. This procedure yields

$$\Phi^{(k,l)}_{\boldsymbol{i},\boldsymbol{j}}(t,t') = \sum_{\pi\in\mathcal{P}(\frac{k+l}{2})} (-1)^{\pi} \prod_{\lambda=1}^{\frac{k+l}{2}} \mathcal{V}^{\boldsymbol{j}}_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)}}(t,t') \tag{59}$$

where $\mathcal{V}$ denotes the case-dependent sums over $\mathcal{U}$

$$\mathcal{V}^{\boldsymbol{j}}_{a,b}(t,t') = \begin{cases} H''_{j_\nu - j_\mu}(2t') & (a,b) = (j_\mu, j_\nu) \\ (-1)^{\nu-1}\,\mathrm{e}^{-\Delta} I_{i_\varepsilon - j_\nu}(\gamma\Delta) + \mathcal{E}^{\boldsymbol{j}}_{i_\varepsilon, j_\nu}(t,t') \qquad \text{for} & (a,b) = (i_\varepsilon, j_\nu) \\ \mathcal{F}^{\boldsymbol{j}}_{i_\varepsilon, i_\delta}(t,t') & (a,b) = (i_\varepsilon, i_\delta) \end{cases} \tag{60}$$

and $\mathcal{E}$, $\mathcal{F}$ are, respectively, the one- and two-dimensional sums

$$\mathcal{E}^{\boldsymbol{j}}_{i_\varepsilon, j_\nu}(t,t') = \sum_{m} \prod_{\lambda=1}^{\dim(\boldsymbol{j})} \operatorname{sgn}(j_\lambda - m)\,\mathrm{e}^{-\Delta} I_{i_\varepsilon - m}(\gamma\Delta) H''_{j_\nu - m}(2t') \tag{61}$$

$$\mathcal{F}^{\boldsymbol{j}}_{i_\varepsilon, i_\delta}(t,t') = \sum_{m,n} \prod_{\lambda=1}^{\dim(\boldsymbol{j})} \operatorname{sgn}(j_\lambda - m)\operatorname{sgn}(j_\lambda - n)\,\mathrm{e}^{-2\Delta} I_{i_\varepsilon - m}(\gamma\Delta) I_{i_\delta - n}(\gamma\Delta) H''_{n-m}(2t'). \tag{62}$$

Equations (59) and (60) are to be read in exactly the same manner as (53) and (54). The sums over m and n in (61) and (62) each run over N different values, so that these expressions are not practical for large system sizes. We show in appendix D, however, that both sums can be reduced to ones involving only a finite number of terms, so that the limit $N \to \infty$ can be taken without problems. In (61) and (62) we have replaced the dimension l of $\boldsymbol{j}$, being the upper limit of the products, by the generic expression $\dim(\boldsymbol{j})$. This is for later convenience. Note that according to (59) and (60) the homogeneous solutions for *arbitrary* order two-time multispin correlation functions can be expressed purely in terms of the functions I, H'', $\mathcal{E}$ and $\mathcal{F}$. Examples of low order homogeneous solutions read

$$\Phi^{(0,2)}_{\cdot,\boldsymbol{j}}(t,t') = H''_{j_2 - j_1}(2t') \tag{63}$$

$$\Phi^{(1,1)}_{i,j}(t,t') = \mathrm{e}^{-\Delta} I_{i-j}(\gamma\Delta) + \mathcal{E}^{j}_{i,j}(t,t') \tag{64}$$

$$\Phi^{(2,2)}_{\boldsymbol{i},\boldsymbol{j}}(t,t') = \mathcal{F}^{\boldsymbol{j}}_{i_1,i_2}(t,t')H''_{j_2-j_1}(2t') - \left[+\mathrm{e}^{-\Delta}I_{i_1-j_1}(\gamma\Delta) + \mathcal{E}^{\boldsymbol{j}}_{i_1,j_1}(t,t')\right]\left[-\mathrm{e}^{-\Delta}I_{i_2-j_2}(\gamma\Delta) + \mathcal{E}^{\boldsymbol{j}}_{i_2,j_2}(t,t')\right] + \left[-\mathrm{e}^{-\Delta}I_{i_1-j_2}(\gamma\Delta) + \mathcal{E}^{\boldsymbol{j}}_{i_1,j_2}(t,t')\right]\left[+\mathrm{e}^{-\Delta}I_{i_2-j_1}(\gamma\Delta) + \mathcal{E}^{\boldsymbol{j}}_{i_2,j_1}(t,t')\right]. \tag{65}$$

Substituting result (59) into (40) finally delivers analytic expressions for the desired two-time correlation functions. We have worked out $C^{(1,1)}(t,t')$, $C^{(2,2)}(t,t')$, $C^{(1,3)}(t,t')$, $C^{(3,1)}(t,t')$ and $C^{(3,3)}(t,t')$ and it appears that a factorization similar to the one which simplified (49) to (51) is generally possible. We do not have a proof for this claim, however, and therefore take (40) together with (59) as our final result, which expresses an arbitrary multispin two-time correlation function as a sum over a finite number of terms. Examples of low order multispin two-time correlations, expressed in terms of homogeneous solutions via (40), read

$$C^{(1,1)}_{i,j}(t,t') = \Phi^{(1,1)}_{i,j}(t,t') \tag{66}$$

$$C^{(2,2)}_{\boldsymbol{i},\boldsymbol{j}}(t,t') = \Phi^{(2,2)}_{\boldsymbol{i},\boldsymbol{j}}(t,t') + H_{i_2-i_1}(2\Delta)\Phi^{(0,2)}_{\cdot,\boldsymbol{j}}(t,t'). \tag{67}$$

In section 7 we present a discussion the correlation functions $C^{(1,1)}(t,t')$ and $C^{(2,2)}(t,t')$. In particular, we demonstrate how expressions (66) and (67) simplify when using the results of appendix D to rewrite the sums (61) and (62).

6. Two-time responses after a quench

Quite analogously to the case of two-time correlations discussed above, two-time multispin response functions $R^{(k,l)}(t,t')$ follow from the general solution (40) with time arguments replaced by $\Delta = t - t'$. The equal time initial conditions $R^{(k,l)}(t',t')$ are given in terms of correlation functions in (15). We consider again a quench from equilibrium as introduced in section 4. Consequently, on all levels k with odd $k+l$ the initial conditions (15) vanish and so do the $R^{(k,l)}(t,t')$ for all $t \geqslant t'$. Thus we immediately focus on $k+l$ even. Without loss of generality we restrict the vector $\boldsymbol{j}$, labelling the sites to which the perturbation couples, to ordered $\boldsymbol{j} \in N(l)$.

Equation (15) comprises a correlation function over the sites $\boldsymbol{i} \cup \boldsymbol{j}^\nu$, where $\boldsymbol{j}^\nu$ is the index vector $\boldsymbol{j}$ with the additional components $j_\nu - 1$ and $j_\nu + 1$ as defined below (15). Since $\boldsymbol{j}$ is fixed we can immediately use the hierarchical property (5) of correlation functions to remove component pairs from $\boldsymbol{j}^\nu$ in cases where $\boldsymbol{j}^\nu \notin N(l+2)$. More precisely, this corresponds to replacing $\boldsymbol{j}^\nu$ in (15) by the 'squashed' index vector

$$\boldsymbol{j}^{\nu,\mathrm{s}} = \begin{cases} (j_1,\ldots,j_{\nu-1}, & j_\nu - 1, & j_\nu, & j_\nu + 1, & j_{\nu+1},\ldots,j_l) & j_{\nu-1} < j_\nu - 1,\ j_\nu + 1 < j_{\nu+1} \\ (j_1,\ldots,j_{\nu-2}, & & j_\nu, & j_\nu + 1, & j_{\nu+1},\ldots,j_l) & j_{\nu-1} = j_\nu - 1,\ j_\nu + 1 < j_{\nu+1} \\ (j_1,\ldots,j_{\nu-1}, & j_\nu - 1, & j_\nu, & & j_{\nu+2},\ldots,j_l) & j_{\nu-1} < j_\nu - 1,\ j_\nu + 1 = j_{\nu+1} \\ (j_1,\ldots,j_{\nu-2}, & & j_\nu, & & j_{\nu+2},\ldots,j_l) & j_{\nu-1} = j_\nu - 1,\ j_\nu + 1 = j_{\nu+1}. \end{cases} \quad \text{for} \tag{68}$$

We denote the dimension of $\boldsymbol{j}^{\nu,\mathrm{s}}$ by $l^{\nu,\mathrm{s}} \in \{l-2, l, l+2\}$. Having made this modification in (15), which guarantees that $\boldsymbol{j}^{\nu,\mathrm{s}} \in N(l^{\nu,\mathrm{s}})$ is ordered, we may now use (53) to write the initial condition $\bar{A}$ corresponding to (15); recall that (53) is the permutationally antisymmetric version of (9). This gives for the permutationally antisymmetrized initial condition for $R^{(k,l)}_{\boldsymbol{i},\boldsymbol{j}}(t,t')$

$$\bar{A}^{(k,l)}_{\boldsymbol{i},\boldsymbol{j}}(t') = \sum_{\mu=1}^{k}\sum_{\nu=1}^{l}\delta_{i_\mu,j_\nu}\left[\left(1-\frac{\gamma^2}{2}\right)\sum_{\pi\in\mathcal{P}(\frac{k+l}{2})}(-1)^{\pi}\prod_{\lambda=1}^{\frac{k+l}{2}}\mathcal{U}^{\boldsymbol{j}}_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)}}(t')\right.$$
$$\left.-\frac{\gamma^2}{2}\sum_{\pi\in\mathcal{P}(\frac{k+l^{\nu,\mathrm{s}}}{2})}(-1)^{\pi}\prod_{\lambda=1}^{\frac{k+l^{\nu,\mathrm{s}}}{2}}\mathcal{U}^{\boldsymbol{j}^{\nu,\mathrm{s}}}_{(\boldsymbol{i}\cup\boldsymbol{j}^{\nu,\mathrm{s}})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j}^{\nu,\mathrm{s}})_{\pi(2\lambda)}}(t')\right]. \tag{69}$$

The fact that the Kronecker deltas in (69) contain components of $\boldsymbol{i}$ does not affect permutational antisymmetry: the only occurrence of the summation variable μ in (69) is in δ_{i_μ,j_ν}. But finite sums are independent of the order in which they are taken, hence permutational antisymmetry is unaffected.

We now note that for each μ,ν the Kronecker delta in (69), enforcing $i_\mu = j_\nu$, allows us to simplify the expressions inside the square brackets. As discussed in the text below equations (53), (54), the products over λ are zero for all pairings that do not contain the pair (i_μ, j_ν). So for each μ,ν it is sufficient to restrict the sums over the ordered pairings in (69) to those that do contain the pair (i_μ, j_ν). For the pairings drawn from $\boldsymbol{i}\cup\boldsymbol{j}$, the corresponding factor $\mathcal{U}$ with index $(a,b) = (i_\mu, j_\nu)$ is $(-1)^{\nu-1}$ according to (54). But for the pairings drawn from $\boldsymbol{i}\cup\boldsymbol{j}^{\nu,\mathrm{s}}$ this factor is $-(-1)^{\nu-1}$ since j_ν is component number $\nu+1$ or $\nu-1$ of $\boldsymbol{j}^{\nu,\mathrm{s}}$ as can be seen from (68). Thus we may rewrite (69) as

$$\bar{A}^{(k,l)}_{\boldsymbol{i},\boldsymbol{j}}(t') = \sum_{\mu=1}^{k}\sum_{\nu=1}^{l}(-1)^{\nu-1}\delta_{i_\mu,j_\nu}\left[\left(1-\frac{\gamma^2}{2}\right)\sum_{\pi\in\mathcal{P}'(\frac{k+l}{2})}(-1)^{\pi}\prod_{\lambda=1}^{\frac{k+l}{2}}\mathcal{U}'^{\boldsymbol{j}}_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)}}(t')\right.$$
$$\left.+\frac{\gamma^2}{2}\sum_{\pi\in\mathcal{P}'(\frac{k+l^{\nu,\mathrm{s}}}{2})}(-1)^{\pi}\prod_{\lambda=1}^{\frac{k+l^{\nu,\mathrm{s}}}{2}}\mathcal{U}'^{\boldsymbol{j}^{\nu,\mathrm{s}}}_{(\boldsymbol{i}\cup\boldsymbol{j}^{\nu,\mathrm{s}})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j}^{\nu,\mathrm{s}})_{\pi(2\lambda)}}(t')\right]. \tag{70}$$

Here $\mathcal{P}'$ emphasizes that we only sum over the pairings that contain (i_μ, j_ν) for each given μ,ν. Since we have pulled the factors $\pm(-1)^{\nu-1}$—contained in each product in (70)—out of the square brackets we have to replace $\mathcal{U}$ by 1 for $(a,b) = (i_\mu, j_\nu)$, and indicated this by writing $\mathcal{U}'$ instead of $\mathcal{U}$.

From the permutationally antisymmetric initial condition (70) that corresponds to (15) one easily obtains the homogeneous solutions for two-time multispin response functions via (28). We use the fact that according to our construction of (70) the entire expression inside the square brackets is independent of i_μ for each fixed μ. So the summation over this index is trivial. The sum over the remaining indices $\boldsymbol{i}\setminus(i_\mu)$ follows in full analogy with the derivation of (59) and consequently we obtain

$$\Phi^{(k,l)}_{\boldsymbol{i},\boldsymbol{j}}(t,t') = \sum_{\mu=1}^{k}\sum_{\nu=1}^{l}(-1)^{\nu-1}\,\mathrm{e}^{-\Delta}I_{i_\mu-j_\nu}(\gamma\Delta)$$
$$\times\left[\left(1-\frac{\gamma^2}{2}\right)\sum_{\pi\in\mathcal{P}'(\frac{k+l}{2})}(-1)^{\pi}\prod_{\lambda=1}^{\frac{k+l}{2}}\mathcal{V}'^{\boldsymbol{j}}_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)}}(t,t')\right.$$
$$\left.+\frac{\gamma^2}{2}\sum_{\pi\in\mathcal{P}'(\frac{k+l^{\nu,\mathrm{s}}}{2})}(-1)^{\pi}\prod_{\lambda=1}^{\frac{k+l^{\nu,\mathrm{s}}}{2}}\mathcal{V}'^{\boldsymbol{j}^{\nu,\mathrm{s}}}_{(\boldsymbol{i}\cup\boldsymbol{j}^{\nu,\mathrm{s}})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j}^{\nu,\mathrm{s}})_{\pi(2\lambda)}}(t,t')\right]. \tag{71}$$

Here $\mathcal{V}' = 1$ for indices $(a,b) = (i_\mu, j_\nu)$ and $\mathcal{V}' = \mathcal{V}$ as given by (60) otherwise. As in the case of two-time correlations we see that the homogeneous solutions for *arbitrary* two-time

multispin response functions can be expressed entirely in terms of I, H'', $\mathcal{E}$ and $\mathcal{F}$ according to (71). At level $k = 0$, the sum over μ in (71) is empty and so we set $\Phi^{(0,l)} = A^{(0,l)} = 0$ as discussed at the end of section 1.3. In appendix E we explicitly demonstrate how the first nontrivial homogeneous solutions (73) and (74) are obtained from the general result (71):

$$\Phi^{(0,2)}_{,\boldsymbol{j}} = 0 \tag{72}$$

$$\Phi^{(1,1)}_{i,j} = \mathrm{e}^{-\Delta} I_{i-j} \left[\left(1 - \frac{\gamma^2}{2} \right) - \frac{\gamma^2}{2} H_2''(2t') \right] \tag{73}$$

$$\begin{aligned}
\Phi^{(2,2)}_{\boldsymbol{i},\boldsymbol{j}} = {} & \mathrm{e}^{-\Delta} I_{i_1-j_1} \left[-\left(1 - \frac{\gamma^2}{2} \right) \left(-\mathrm{e}^{-\Delta} I_{i_2-j_2} + \mathcal{E}^{(j_1,j_2)}_{i_2,j_2} \right) + \frac{\gamma^2}{2} \left(+\mathrm{e}^{-\Delta} I_{i_2-j_1+1} + \mathcal{E}^{(j_1-1,j_1)}_{i_2,j_1-1} \right) \right] \\
& + \mathrm{e}^{-\Delta} I_{i_2-j_1} \left[+\left(1 - \frac{\gamma^2}{2} \right) \left(-\mathrm{e}^{-\Delta} I_{i_1-j_2} + \mathcal{E}^{(j_1,j_2)}_{i_1,j_2} \right) \right. \\
& \left. - \frac{\gamma^2}{2} \left(+\mathrm{e}^{-\Delta} I_{i_1-j_1+1} + \mathcal{E}^{(j_1-1,j_1)}_{i_1,j_1-1} \right) \right] \\
& - \mathrm{e}^{-\Delta} I_{i_1-j_2} \left[+\left(1 - \frac{\gamma^2}{2} \right) \left(+\mathrm{e}^{-\Delta} I_{i_2-j_1} + \mathcal{E}^{(j_1,j_2)}_{i_2,j_1} \right) \right. \\
& \left. - \frac{\gamma^2}{2} \left(-\mathrm{e}^{-\Delta} I_{i_2-j_2-1} + \mathcal{E}^{(j_2,j_2+1)}_{i_2,j_2+1} \right) \right] \\
& - \mathrm{e}^{-\Delta} I_{i_2-j_2} \left[-\left(1 - \frac{\gamma^2}{2} \right) \left(+\mathrm{e}^{-\Delta} I_{i_1-j_1} + \mathcal{E}^{(j_1,j_2)}_{i_1,j_1} \right) \right. \\
& \left. + \frac{\gamma^2}{2} \left(-\mathrm{e}^{-\Delta} I_{i_1-j_2-1} + \mathcal{E}^{(j_2,j_2+1)}_{i_1,j_2+1} \right) \right].
\end{aligned} \tag{74}$$

To save space we have omitted the arguments of the functions I and $\mathcal{E}$ which are obviously $\gamma\Delta$ and (t, t'), respectively. Expression (74) for $\Phi^{(2,2)}$ only holds if $\boldsymbol{j}$ is of the form $\boldsymbol{j} = (j, j+1)$; for $j_1 + 1 < j_2$ the result (71) gives a more complicated form.

As usual, actual two-time response functions may be expressed in terms of homogeneous solutions (71) via (40). It is not clear to us whether a factorization of the combined expression (40), (71) is possible. So we state our results explicitly in terms of homogeneous solutions, as for instance

$$R^{(1,1)}_{i,j}(t, t') = \Phi^{(1,1)}_{i,j}(t, t') \tag{75}$$

$$R^{(2,2)}_{\boldsymbol{i},\boldsymbol{j}}(t, t') = \Phi^{(2,2)}_{\boldsymbol{i},\boldsymbol{j}}(t, t') + H_{i_2-i_1}(2\Delta)\Phi^{(0,2)}_{,\boldsymbol{j}}(t, t'). \tag{76}$$

The responses $R^{(1,1)}(t, t')$ and $R^{(2,2)}(t, t')$ are discussed further in the subsequent section. In particular, we show that the identities for $\mathcal{E}$ given in appendix D yield a simplification of the expression (76).

7. Simple examples of two-time functions

In this section we illustrate the procedure of extracting explicit expressions for multispin two-time correlation and response functions from the general solutions derived in sections 5 and 6. For some of the examples we also give a discussion of their implication for the physics of the Glauber–Ising chain.

The simplest possible two-time correlation functions comprise only one spin at each time. By combining (64), (66) and rewriting the sum $\mathcal{E}$ using (D6) we obtain

$$C_{i,j}^{(1,1)}(t,t') = \langle \sigma_i(t)\sigma_j(t')\rangle = \mathrm{e}^{-\Delta} I_{i-j}(\gamma\Delta) + \tilde{H}''_{j-i}(\Delta, 2t'). \tag{77}$$

Expressions for the functions $I_n(x)$, $\tilde{H}''_n(t_1,t_2)$ are given in (B1), (C11). Since the two-time correlations (77) are the solutions of (8) at level $k=1$, the set $Q = Q_\mathrm{o}$ must be used in (B1). Result (77) applies for the finite $1d$ Glauber–Ising model quenched from an equilibrium state at some $T_\mathrm{i} > 0$ to an arbitrary temperature $T \geqslant 0$ at time $t = 0$. According to the discussion of the functions I and H in appendices B and C, taking the thermodynamic limit $N \to \infty$ and quenching from a random initial configuration $T_\mathrm{i} = \infty$ just amounts to replacing I_{j-i} and $\tilde{H}''_{j-i}$ in (77) with I_{j-i} and $\tilde{\mathrm{H}}_{j-i}$, respectively. Writing out $\tilde{\mathrm{H}}_{j-i}(\Delta, 2t')$ explicitly using (C11) we thus have

$$C_{i,j}^{(1,1)}(t,t') = \mathrm{e}^{-\Delta}\mathrm{I}_{i-j}(\gamma\Delta) + \mathrm{sgn}(j-i)\mathrm{H}_{j-i}(\Delta, 2t') + \gamma\int_0^\Delta \mathrm{d}\tau\, \mathrm{e}^{-\tau}\mathrm{I}_{j-i}(\gamma\tau)\mathrm{H}_1(\Delta-\tau, 2t'). \tag{78}$$

If we additionally quench the system to $T = 0$ then (77) together with the zero temperature formula (C23) for $\tilde{\mathrm{H}}_n(t_1,t_2)$ simply yields

$$C_{i,j}^{(1,1)}(t,t') = \mathrm{e}^{-(t+t')}\left\{\mathrm{I}_{i-j}(t+t') + \int_0^{2t'} \mathrm{d}\tau\, \mathrm{I}_{i-j}(t+t'-\tau)\,[\mathrm{I}_0 + \mathrm{I}_1](\tau)\right\}. \tag{79}$$

Here we have introduced the short hand $[\cdot](x)$ to indicate that all functions contained in the square bracket have the same argument x. The functions $\mathrm{I}_n(x)$ in (78), (79) are the modified Bessel functions (B2).

Similarly we obtain the two-time spin response functions to a local magnetic field h_j in the non-equilibrium state after the quench—scaled by T according to our definition in section 1.3–by putting together (73), (75)

$$R_{i,j}^{(1,1)}(t,t') = T\left.\frac{\delta\langle\sigma_i(t)\rangle}{\delta h_j(t')}\right|_{h_j=0} = \mathrm{e}^{-\Delta} I_{i-j}(\gamma\Delta)\left[1 - \frac{\gamma^2}{2}(1 + H''_2(2t'))\right]. \tag{80}$$

$I_n(x)$ and $H''_n(t)$ are given in (B1) (with $Q = Q_\mathrm{o}$) and (52). Taking the thermodynamic limit $N \to \infty$ and letting $T_\mathrm{i} = \infty$ turns (80) into

$$R_{i,j}^{(1,1)}(t,t') = \mathrm{e}^{-\Delta}\mathrm{I}_{i-j}(\gamma\Delta)\left[1 - \frac{\gamma^2}{2}(1 + \mathrm{H}_2(2t'))\right]. \tag{81}$$

For the zero temperature quench (81) may be simplified further by using (C17), giving

$$R_{i,j}^{(1,1)}(t,t') = \tfrac{1}{2}\mathrm{e}^{-(t+t')}\mathrm{I}_{i-j}(\Delta)\,[\mathrm{I}_0 + 2\mathrm{I}_1 + \mathrm{I}_2]\,(2t'). \tag{82}$$

The exact solutions (78), (81) agree, as they should, with their temporal Laplace (with respect to Δ, t') and spatial Fourier (with respect to $n = i-j$) transforms as derived in [10]. While the latter are surprisingly simple for the functions (78) and (81), a corresponding transformation of higher order functions is virtually impossible. The correlations (78) and a modified version of the responses (81), slightly differing in the way the transition rates in the presence of a field are defined, were also studied in [11]; the long-time scaling form of the correlation functions can equivalently be found using a picture of annihilating random walkers [17].

Both publications [10, 11] considered the non-equilibrium fluctuation–dissipation relations associated with the auto-correlation and response functions. For any pair of correlation and response functions one can define a fluctuation–dissipation theorem (FDT) 'violation' factor X via

$$R(t,t') = X(t,t')\frac{\partial}{\partial t'}C(t,t') \tag{83}$$

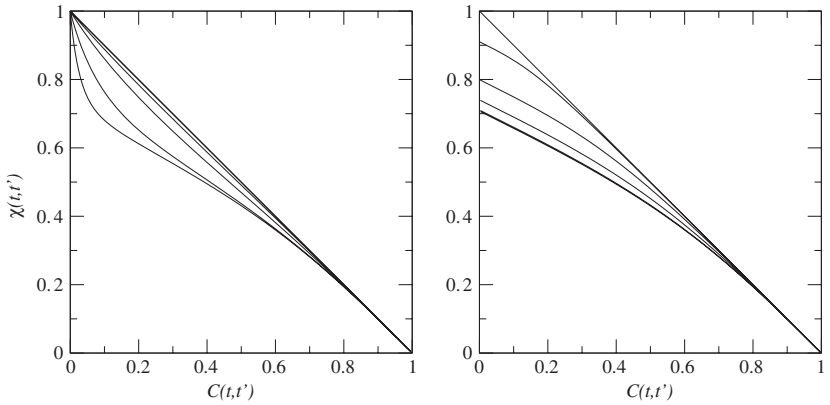

Figure 2. FD plots of the local spin–spin correlation and response functions, for fixed t' (left) and fixed t (right). The values of $t'/\tau_{\rm eq}$ are $t'/\tau_{\rm eq} = 10^{-3}, 10^{-2}, 10^{-1}, 10^{-1/2}, 10^{0}, 10^{1}$ from bottom to top on the left; the same values are used for $t/\tau_{\rm eq}$ on the right. See the text for discussion.

so that $X(t,t') = 1$ corresponds to the usual equilibrium FDT (recall that our response function contains an extra factor of T; in the conventional definition, (83) would have X/T on the rhs instead of X). Under appropriate conditions, T/X can then be interpreted as an effective temperature $T_{\rm eff}$; this interpretation is well-established for mean field spin glass models but its extension away from mean field remains open and has prompted much recent work [18–20].

In [12], we have given a detailed discussion of the FDT violations that result for non-local correlation and response functions, extending the work of [10, 11] for the local quantities. Here we therefore only illustrate one aspect of our results, by considering the local quantities obtained for $i = j$ from (78), (81) for a quench to nonzero temperature. When all times t, t' and $\tau_{\rm eq} = 1/(1-\gamma)$ are large, one expects results that depend only on the time ratios t/t' and $t'/\tau_{\rm eq}$; this is indeed what we find, in agreement with the results given in [11]. A convenient way of representing the results is a parametric fluctuation–dissipation (FD) plot of the integrated response function against the correlation function; the former is defined as

$$\chi(t,t') = \int_{t'}^{t} \mathrm{d}\tau\, R(t,\tau) \tag{84}$$

and gives the response to a field switched on at t' and held constant after that. One notes from (83) and $R(t,t') = -(\partial/\partial t')\chi(t,t')$ that the negative slope of such a plot is guaranteed to give $X(t,t')$ only if t' is used as the curve parameter while t is held fixed [21]. This gives the FD plots in figure 2(right); on the left we show the FD plot reported in [11], which was produced with the opposite convention of keeping t' fixed and varying t along the curves. We see that the plot on the right produces a physically far more plausible representation of the crossover from the aging regime $t' \ll \tau_{\rm eq}$ to equilibrium $t' \gg \tau_{\rm eq}$; in particular, one reads off that the FDT violation factor $X(t,t')$ is always $\leqslant 1$ and its value in the limit $C \to 0$ evolves smoothly from $1/2$ to 1 as the ratio $t/\tau_{\rm eq}$ grows from 0 to ∞.

Among the higher order correlation functions, one of primary interest is the domain wall correlation. The domain wall indicator (or defect variable) $n_i = \frac{1}{2}(1 - \sigma_i\sigma_{i+1}) \in \{0,1\}$ measures the presence ($n_i = 1$) of a domain wall between sites i and $i+1$. It is a well-known fact [6–8] that Glauber dynamics of the (unmagnetised) spin system $\boldsymbol{\sigma}$ corresponds to the diffusion limited reaction process $A + A \rightleftharpoons \emptyset$ in terms of $\boldsymbol{n} = (n_1, \ldots, n_N)$; here we interpret n_i as the particle occupation number of site i. The relevant rates follow immediately from (2): the transitions $\ldots \uparrow\downarrow\downarrow \ldots \rightleftharpoons \ldots \uparrow\uparrow\downarrow \ldots$ for spins, where $\uparrow$ and $\downarrow$ symbolize $\sigma_i = +1$ and

$\sigma_i = -1$, respectively, occur with rate $1/2$ and correspond to diffusion $\ldots 10 \ldots \rightleftharpoons \ldots 01 \ldots$ of particles. The process $\ldots \uparrow\downarrow\uparrow \ldots \rightleftharpoons \ldots \uparrow\uparrow\uparrow \ldots$, on the other hand, which maps onto $\ldots 11 \ldots \rightleftharpoons \ldots 00 \ldots$, gives the rate $(1+\gamma)/2$ for particle annihilation and the $(1-\gamma)/2$ for pair creation. For zero temperature dynamics, where $\gamma = 1$, this obviously reduces to diffusion limited pair annihilation (DLPA). In the context of the Glauber–Ising model, understanding the dynamics of domain walls or defects is crucial for, e.g. explaining the coarsening process after a quench. Here our general solutions for multispin two-time correlation and response functions given in the preceding sections are the key for a comprehensive analysis. Due to the mapping to diffusion reaction processes, our results simultaneously describe multi-particle two-time correlation and response functions in the latter systems.

As a simple example we now consider the two-time connected correlation function between two domain walls, or equivalently particles, $C_{i-j}(t,t') = 4[\langle n_i(t) n_j(t')\rangle - \langle n_i(t)\rangle\langle n_j(t')\rangle]$. In terms of spins this gives

$$C_{i-j}(t,t') = \langle \sigma_i(t)\sigma_{i+1}(t)\sigma_j(t')\sigma_{j+1}(t')\rangle - \langle \sigma_i(t)\sigma_{i+1}(t)\rangle\langle \sigma_j(t')\sigma_{j+1}(t')\rangle. \tag{85}$$

Now consider the coarsening dynamics after quenching the system from an equilibrium state at $T_\mathrm{i} > 0$ to some temperature $T \geqslant 0$. The two-spin correlations in (85) follow immediately from (51), i.e. $\langle \sigma_i(t)\sigma_{i+1}(t)\rangle = H_1''(2t)$. The non-trivial term in (85) is the two-time four-spin correlation function. Combining the results (63), (65) and (67) we may write any such correlation as

$$\begin{aligned} C^{(2,2)}_{\boldsymbol{i},\boldsymbol{j}}(t,t') = {} & \big[H_{i_2-i_1}(2\Delta) + \mathcal{F}^{\boldsymbol{j}}_{i_1,i_2}(t,t')\big] H''_{j_2-j_1}(2t') \\ & - \big[+\mathrm{e}^{-\Delta} I_{i_1-j_1}(\gamma\Delta) + \mathcal{E}^{\boldsymbol{j}}_{i_1,j_1}(t,t')\big]\big[-\mathrm{e}^{-\Delta} I_{i_2-j_2}(\gamma\Delta) + \mathcal{E}^{\boldsymbol{j}}_{i_2,j_2}(t,t')\big] \\ & + \big[-\mathrm{e}^{-\Delta} I_{i_1-j_2}(\gamma\Delta) + \mathcal{E}^{\boldsymbol{j}}_{i_1,j_2}(t,t')\big]\big[+\mathrm{e}^{-\Delta} I_{i_2-j_1}(\gamma\Delta) + \mathcal{E}^{\boldsymbol{j}}_{i_2,j_1}(t,t')\big]. \end{aligned} \tag{86}$$

To obtain the two-time correlation in the first term of (85) we set $\boldsymbol{i} = (i, i+1)$ and $\boldsymbol{j} = (j, j+1)$. For these index vectors and bearing in mind that $H''_n = -H''_{-n}$, we obtain from (D3) and (D8) the following representations for the sums $\mathcal{E}$, $\mathcal{F}$ in (86)

$$\begin{aligned} \mathcal{E}^{\boldsymbol{j}}_{i_1,j_1}(t,t') &= H''_{j-i}(\Delta, 2t') + \mathrm{e}^{-\Delta} I_{i-j-1}(\gamma\Delta) H''_1(2t') \\ \mathcal{E}^{\boldsymbol{j}}_{i_1,j_2}(t,t') &= H''_{j-i+1}(\Delta, 2t') - \mathrm{e}^{-\Delta} I_{i-j}(\gamma\Delta) H''_1(2t') \\ \mathcal{E}^{\boldsymbol{j}}_{i_2,j_1}(t,t') &= H''_{j-i-1}(\Delta, 2t') + \mathrm{e}^{-\Delta} I_{i-j}(\gamma\Delta) H''_1(2t') \\ \mathcal{E}^{\boldsymbol{j}}_{i_2,j_2}(t,t') &= H''_{j-i}(\Delta, 2t') - \mathrm{e}^{-\Delta} I_{i-j+1}(\gamma\Delta) H''_1(2t') \\ \mathcal{F}^{\boldsymbol{j}}_{i_1,i_2}(t,t') &= H''_1(2\Delta, 2t') \\ &\quad - \mathrm{e}^{-\Delta}[I_{i-j}(\gamma\Delta) H''_{i-j+1}(\Delta, 2t') - I_{i-j+1}(\gamma\Delta) H''_{i-j}(\Delta, 2t')] \\ &\quad - \mathrm{e}^{-\Delta}[I_{i-j-1}(\gamma\Delta) H''_{i-j}(\Delta, 2t') - I_{i-j}(\gamma\Delta) H''_{i-j-1}(\Delta, 2t')] \\ &\quad + \mathrm{e}^{-2\Delta} I_{i-j}(\gamma\Delta) I_{i-j}(\gamma\Delta) H''_1(2t') - \mathrm{e}^{-2\Delta} I_{i-j-1}(\gamma\Delta) I_{i-j+1}(\gamma\Delta) H''_1(2t'). \end{aligned}$$

Substituting these expressions into (86), using the identity (C10) to rewrite $H_n(2\Delta) + H''_n(2\Delta, 2t') = H''_n(2t)$ and cancelling terms gives

$$\begin{aligned} C^{(2,2)}_{\boldsymbol{i},\boldsymbol{j}}(t,t') = {} & H''_1(2t) H''_1(2t') \\ & + [\mathrm{e}^{-\Delta} I_{i-j}(\gamma\Delta) + H''_{j-i}(\Delta, 2t')][\mathrm{e}^{-\Delta} I_{i-j}(\gamma\Delta) - H''_{j-i}(\Delta, 2t')] \\ & - [\mathrm{e}^{-\Delta} I_{i-j-1}(\gamma\Delta) - H''_{j-i+1}(\Delta, 2t')][\mathrm{e}^{-\Delta} I_{i-j+1}(\gamma\Delta) + H''_{j-i-1}(\Delta, 2t')]. \end{aligned} \tag{87}$$

Note that the first term on the rhs of (87), $H''_1(2t) H''_1(2t')$, cancels with $\langle \sigma_i(t)\sigma_{i+1}(t)\rangle\langle \sigma_j(t')\sigma_{j+1}(t')\rangle$ when we substitute into (85). The correlations (87) are those

in a finite ring of spins; since (87) was obtained as the solutions of (8) on level $k = 2$ the set $Q = Q_e$ has to be used in the representation (B1) for $I_n(x)$. The functions $H_n''(t_1, t_2)$ are given in (C9). Now we focus on the quench from a random initial state $T_i = \infty$ and take the thermodynamic limit $N \to \infty$, thus replacing $I_n(x)$ and $H_n''(t_1, t_2)$ in (87) by $\mathrm{I}_n(x)$ and $\mathrm{H}_n(t_1, t_2)$, respectively. Substitution into (85) then yields the two-time connected defect correlation function

$$C_n(t, t') = [\mathrm{e}^{-\Delta}\mathrm{I}_n(\gamma\Delta) + \mathrm{H}_n(\Delta, 2t')][\mathrm{e}^{-\Delta}\mathrm{I}_n(\gamma\Delta) - \mathrm{H}_n(\Delta, 2t')] - [\mathrm{e}^{-\Delta}\mathrm{I}_{n+1}(\gamma\Delta) - \mathrm{H}_{n+1}(\Delta, 2t')][\mathrm{e}^{-\Delta}\mathrm{I}_{n-1}(\gamma\Delta) + \mathrm{H}_{n-1}(\Delta, 2t')]. \tag{88}$$

For the $T = 0$ quench (88) may be simplified further using (C17)

$$C_n(t, t') = \mathrm{e}^{-(t+t')}[\mathrm{I}_{n-1} - \mathrm{I}_{n+1}](t + t')\mathrm{H}_n(t - t', 2t') + \mathrm{e}^{-2t}\mathrm{I}_n(t - t') \times [\mathrm{I}_{n-1} + 2\mathrm{I}_n + \mathrm{I}_{n+1}](t + t') - \mathrm{e}^{-2(t+t')}[(\mathrm{I}_{n-1} + \mathrm{I}_n)(\mathrm{I}_n + \mathrm{I}_{n+1})](t + t'). \tag{89}$$

For small n the functions $\mathrm{H}_n(t_1, t_2)$ in (89) may be expressed purely in terms of modified Bessel functions $\mathrm{I}_n(x)$ via the recursion (C17). For large n, on the other hand, it is more convenient to use the representation (C4), with $I_n(x)$ replaced by $\mathrm{I}_n(x)$ and $\gamma = 1$.

We have used the zero temperature result (89) in [12] to study the non-equilibrium fluctuation–dissipation relations for defect observables n_i. Beyond this, we are not aware of any expressions equivalent to, e.g. (88) in the literature. As explained above, our results are of interest not only for the coarsening dynamics of the Glauber–Ising spin chain, but also directly give the corresponding particle–particle correlation and response functions for the associated diffusion reaction process.

From (88) we can, for instance, derive an exact expression for the equilibrium domain-wall autocorrelation. This function corresponds to (four times) the particle autocorrelation in a diffusion limited reaction process $A + A \rightleftharpoons \emptyset$ at equilibrium. It follows from (88) by setting $n = 0$ and taking the limit $t' \to \infty$ with $\Delta = t - t'$ fixed. This gives, using the representation (C4) for $\mathrm{H}_n(\Delta, 2t')$,

$$C_{\mathrm{eq}}(\Delta) = \lim_{t'\to\infty} C_0(\Delta + t', t') = \mathrm{e}^{-2\Delta}\mathrm{I}_0^2(\gamma\Delta) - \left\{\mathrm{e}^{-\Delta}\mathrm{I}_1(\gamma\Delta) - \frac{\gamma}{2}\int_\Delta^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau}[\mathrm{I}_0 - \mathrm{I}_2](\gamma\tau)\right\}^2. \tag{90}$$

Expanding (90) asymptotically for $\Delta, \tau_{\mathrm{eq}} \to \infty$ at fixed $\Delta/\tau_{\mathrm{eq}}$ gives the scaling forms

$$1 \ll \Delta \ll \tau_{\mathrm{eq}} : C_{\mathrm{eq}}(\Delta) \sim \frac{2}{\sqrt{\pi\,\tau_{\mathrm{eq}}\Delta}} \tag{91}$$

$$1 \ll \tau_{\mathrm{eq}} \ll \Delta : C_{\mathrm{eq}}(\Delta) \sim \frac{\tau_{\mathrm{eq}}}{\pi\,\Delta^2}\,\mathrm{e}^{-2\Delta/\tau_{\mathrm{eq}}}. \tag{92}$$

Plots of (90) for various τ_{eq} are shown in figure 3. In the regime $\Delta \ll \tau_{\mathrm{eq}}$ the particles may be considered as an ensemble of independent random walkers. Hence the decline of $C_{\mathrm{eq}}(\Delta)$ corresponds to the return probability of a random walker. In the opposite regime, however, the particles are extremely likely to have been annihilated and replaced by new ones via the process $A + A \leftharpoondown \emptyset$. Since these new particles are uncorrelated with the original ones, the connected correlation vanishes.

For comparison we also consider the non-equilibrium case of diffusion limited pair-annihilation $A + A \to \emptyset$, for which the correlation function is given by the zero temperature formula (89). Setting $n = 0$ again yields an exact expression for the particle autocorrelation

$$C_0(t, t') = 2\mathrm{e}^{-2t}\mathrm{I}_0(\Delta)[\mathrm{I}_0 + \mathrm{I}_1](t + t') - \mathrm{e}^{-2(t+t')}[\mathrm{I}_0 + \mathrm{I}_1]^2(t + t'). \tag{93}$$

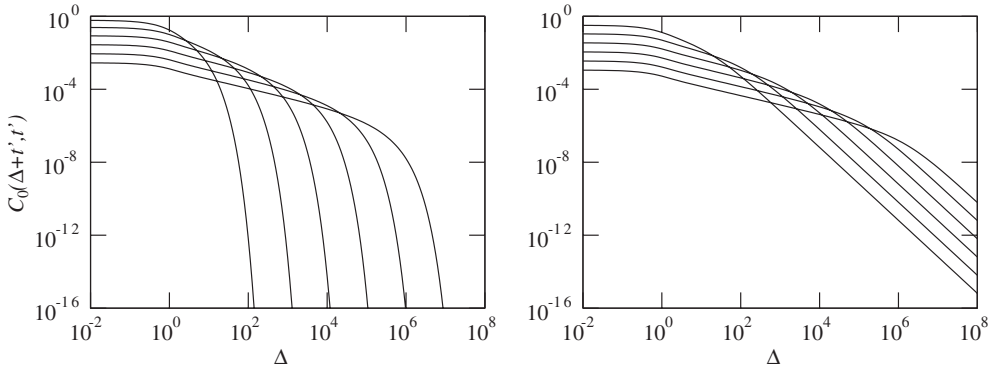

Figure 3. Connected domain wall autocorrelation functions $C_0(t,t') = 4[\langle n_i(t)n_i(t')\rangle - \langle n_i(t)\rangle\langle n_i(t')\rangle]$ in equilibrium (left), i.e. $t' \to \infty$, and for the coarsening dynamics after a zero temperature quench at $t = 0$ from a random initial state (right). The different curves correspond to decreasing temperature (left) and increasing system 'age' t' (right) from top to bottom at small $\Delta = t - t'$. The plots are obtained by numerical evaluation of equations (90), (93) for $\tau_{\rm eq}, t' = 10^1, 10^2, \ldots, 10^6$ and are discussed in the text.

In (93) the time t' plays a role analogous to $\tau_{\rm eq}$ in (90). So we expand for $\Delta, t' \to \infty$ with Δ/t' fixed. This gives the scaling forms

$$1 \ll \Delta \ll t' : C_0(t,t') \sim \frac{2}{\pi\sqrt{2t'\Delta}} \tag{94}$$

$$1 \ll t' \ll \Delta : C_0(t,t') \sim \frac{2t'}{\pi\Delta^2}. \tag{95}$$

Plots of (93) are also shown in figure 3. For $\Delta \ll t'$ we have a situation that is exactly analogous to the equilibrium case, with $\tau_{\rm eq}$ replaced by $2\pi t'$. In the opposite regime $\Delta \gg t'$, however, the correlation function decays much more slowly, now due to the annihilation of particles that are very far apart rather than, as in equilibrium, the creation of new particle pairs. We discuss in [12] that the scaling form (95) is nontrivial and, for instance, gives rise to unanticipated nonequilibrium fluctuation–dissipation relations in the Glauber–Ising model.

The counterpart of the two-time connected domain wall correlation function (85) is the two-time domain wall response function to local bond perturbations $h_{j,j+1}$ defined by

$$R_{i-j}(t,t') = T\left.\frac{\delta\langle\sigma_i(t)\sigma_{i+1}(t)\rangle}{\delta h_{j,j+1}(t')}\right|_{h_{j,j+1}=0}. \tag{96}$$

In full analogy with the correlations we obtain $R_n(t,t')$ from (74), (76) and (D3), with $\boldsymbol{i} = (i, i+1)$ and $\boldsymbol{j} = (j, j+1)$. For $N \to \infty$ and a quench from a random initial state $T_{\rm i} = \infty$ we find

$$\begin{aligned} R_n(t,t') = {} & {\rm e}^{-\Delta}{\rm I}_n\left[-\left(1-\frac{\gamma^2}{2}\right)(-{\rm e}^{-\Delta}{\rm I}_n + {\rm H}_n) + \frac{\gamma^2}{2}(+{\rm e}^{-\Delta}{\rm I}_{n-2} + {\rm H}_{n-2})\right] \\ & + {\rm e}^{-\Delta}{\rm I}_{n-1}\left[+\left(1-\frac{\gamma^2}{2}\right)(-{\rm e}^{-\Delta}{\rm I}_{n+1} + {\rm H}_{n+1}) - \frac{\gamma^2}{2}(+{\rm e}^{-\Delta}{\rm I}_{n-1} + {\rm H}_{n-1})\right] \\ & - {\rm e}^{-\Delta}{\rm I}_{n+1}\left[+\left(1-\frac{\gamma^2}{2}\right)(+{\rm e}^{-\Delta}{\rm I}_{n-1} + {\rm H}_{n-1}) - \frac{\gamma^2}{2}(-{\rm e}^{-\Delta}{\rm I}_{n+1} + {\rm H}_{n+1})\right] \\ & - {\rm e}^{-\Delta}{\rm I}_n\left[-\left(1-\frac{\gamma^2}{2}\right)(+{\rm e}^{-\Delta}{\rm I}_n + {\rm H}_n) + \frac{\gamma^2}{2}(-{\rm e}^{-\Delta}{\rm I}_{n+2} + {\rm H}_{n+2})\right]. \end{aligned} \tag{97}$$

All I_n in (97) have argument $\mathrm{I}_n(\gamma\Delta)$ and all $\mathrm{H}_n = \mathrm{H}_n(\Delta, 2t')$. For the $T = 0$ quench we may use the recursion (C17) for $\mathrm{H}_n(t_1, t_2)$ to simplify this equation *drastically*. It turns out that the result may be written as

$$R_n(t, t') = \frac{\partial}{\partial t'}\{\mathrm{e}^{-2t}\mathrm{I}_n(t - t')[\mathrm{I}_{n-1} + 2\mathrm{I}_n + \mathrm{I}_{n+1}](t + t')\}. \tag{98}$$

The functions C_n (88) and R_n (97) are rather special: for other index values, e.g. $(i, i+2)$, or in particular for higher orders k, l the expressions become rather more complicated and are most efficiently evaluated using symbolic software. As above, simplifications occur for $N \to \infty$ and for a quench from $T_\mathrm{i} = \infty$ to $T = 0$. Any two-time multispin correlation and response function $C^{(k,l)}(t, t')$ and $R^{(k,l)}(t, t')$ with k and l even can then be expressed purely in terms of modified Bessel functions $\mathrm{I}_n(x)$, due to (C17), (D3) and (D8). It is clear from (C17), of course, that the number of $\mathrm{I}_n(x)$ in the result grows with the index range covered by $\boldsymbol{i}, \boldsymbol{j}$.

8. Conclusions

We have presented a new approach for solving the full hierarchy of differential equations for mulitspin correlation and response functions in the finite one-dimensional Glauber–Ising model. Our result is the most explicit representation for these functions. The known results for equilibrium and dynamical correlation functions as well as two-time spin–spin correlation and response functions after a quench are easily recovered from our solution. Beyond that, however, we have derived closed expressions for arbitrary mutispin two-time correlation and response functions for the dynamics after a quench. We found that any such quantity can be expressed purely in terms of the four functions I, H'', $\mathcal{E}$ and $\mathcal{F}$, regardless of the number of spins involved.

Our results for the two-time correlation and response functions formed the basis for the study [12], where we gave a comprehensive analysis of the nonequilibrium fluctuation–dissipation relations in the Glauber–Ising model at zero temperature. A number of other applications of our results can be envisaged. For example, based on the general expression for two-spin two-time correlations, we are currently studying the existence of dynamical hereogeneities in the Glauber–Ising model [22].

Finally, we have emphasized the link between the Glauber–Ising chain and diffusion–reaction processes. In particular, two-time multi-particle correlation and response functions for a one-dimensional diffusion-limited annihilation process with appropriate rates follow immediately from our results. Using the exact mapping between diffusion–annihilation and diffusion–coagulation processes [23], two-time solutions also follow for the latter processes. We are currently exploring this link, with the aim of e.g. obtaining exact scaling formulae for two-time correlation and response functions in the one-spin facilitated Fredrickson–Andersen model [24] at low temperatures. These should help to clarify the meaning and extent of the apparent trivial equilibrium fluctuation–dissipation behaviour found in numerical simulations [25].

Acknowledgments

We acknowledge financial support from the Austrian Academy of Sciences, the Wilhelm Macke Foundation and EPSRC Grant no 00800822 (PM), and the Nuffield Foundation Grant no NAL/00361/G (PS).

Appendix A. Projection of the Green function

Here we derive the identity

$$\sum_{\boldsymbol{j}\in N(k)} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}}(t) b^{(k)}_{\boldsymbol{j},\boldsymbol{j}'} = \sum_{1\leqslant\mu<\nu\leqslant k} (-1)^{\nu-\mu-1} \left(\frac{\mathrm{d}}{\mathrm{d}t} H_{i_\nu - i_\mu}(2t)\right) G^{(k-2)}_{\boldsymbol{i}\setminus(i_\mu,i_\nu),\boldsymbol{j}'}(t) \quad \text{(A1)}$$

that is used in section 2.2 for the inductive proof of (32). The functions $G^{(k)}(t)$ and $H_n(t)$ are given in (30) and (C1), respectively, and $\boldsymbol{i} \in N(k)$, $\boldsymbol{j}' \in N(k-2)$ are ordered indices. The matrix $b^{(k)}$, accounting for links between levels k and $k-2$ in the hierarchy, is discussed in the text below (17) and may be written as

$$b^{(k)}_{\boldsymbol{j},\boldsymbol{j}'} = \gamma \sum_{\eta=1}^{k-1} \delta_{j_{\eta+1}, j_\eta+1} \delta_{\boldsymbol{j}\setminus(j_\eta, j_{\eta+1}),\boldsymbol{j}'} + \gamma \delta_{j_1+N, j_k+1} \delta_{\boldsymbol{j}\setminus(j_1,j_k),\boldsymbol{j}'}. \quad \text{(A2)}$$

Now we focus on (A1): for any fixed $\boldsymbol{j}'$ the summation variable $\boldsymbol{j}$ on the lhs of (A1) must assume the particular values $\boldsymbol{j} = \boldsymbol{j}_{\eta,\ell}$ in order to hit a nonzero matrix element of $b^{(k)}$, where

$$\begin{aligned} &1 \leqslant \eta \leqslant k-1 : \boldsymbol{j}_{\eta,\ell} = (j'_1, \ldots, j'_{\eta-1}, \ell, \ell+1, j'_\eta, \ldots, j'_{k-2}) \\ &\eta = k : \boldsymbol{j}_{k,N} = (1, j'_1, \ldots, j'_{k-2}, N). \end{aligned} \quad \text{(A3)}$$

The constraint $\boldsymbol{j}_{\eta,\ell} \in N(k)$ restricts the range of the parameters η and ℓ. If, for instance, $2 \leqslant \eta \leqslant k-2$ the conditions $j'_{\eta-1} < \ell$ and $\ell+1 < j'_\eta$ must be satisfied. So only if $j'_{\eta-1} \leqslant j'_\eta - 3$ is there enough 'space' to insert the indices $(\ell, \ell+1)$ and retain strict ordering. Otherwise the corresponding vector $\boldsymbol{j}_{\eta,\ell}$ must be omitted. Nevertheless we may write

$$\frac{1}{\gamma} \sum_{\boldsymbol{j}\in N(k)} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}}(t) b^{(k)}_{\boldsymbol{j},\boldsymbol{j}'} = \sum_{\ell=1}^{j'_1-1} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}_{1,\ell}}(t) + \sum_{\eta=2}^{k-2} \sum_{\ell=j'_{\eta-1}}^{j'_\eta-1} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}_{\eta,\ell}}(t) + \sum_{\ell=j'_{k-2}}^{N-1} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}_{k-1,\ell}}(t) + G^{(k)}_{\boldsymbol{i},\boldsymbol{j}_{k,N}}(t). \quad \text{(A4)}$$

Note that in contrast to what we have just said we insert the indices $(\ell, \ell+1)$ between $j'_{\eta-1}$ and j'_η in (A4) even if $j'_{\eta-1} = j'_\eta - 1$. This is possible because the Green function $G^{(k)}$ from (30) is identically zero if $\boldsymbol{j}_{\eta,\ell}$ contains an index pair, as follows from its permutational antisymmetry discussed after (30). We may use permutational antisymmetry of $G^{(k)}$ once more to replace in (A4) all $\boldsymbol{j}_{\eta,\ell}$ with $1 \leqslant \eta < k$ by $\boldsymbol{j}_{1,\ell}$ since the corresponding permutations involve an even number of transpositions. The η-dependencies in (A4) via Green's functions then drop out and we can combine the sums

$$\sum_{\boldsymbol{j}\in N(k)} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}}(t) b^{(k)}_{\boldsymbol{j},\boldsymbol{j}'} = \gamma \sum_{\ell=1}^{N-1} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}_{1,\ell}}(t) + \gamma G^{(k)}_{\boldsymbol{i},\boldsymbol{j}_{k,N}}(t) = \gamma \sum_{\ell=1}^{N} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}_{1,\ell}}(t). \quad \text{(A5)}$$

The second equality in (A5) follows when using permutational antisymmetry of $G^{(k)}$ and N-(anti)-periodicity (B9) of the functions $I_n(x)$ contained in $G^{(k)}$. The reasoning applied is in full analogy with the discussion of $\bar{\Phi}^{(k)}$ in section 2.1. Before proceeding, we discuss briefly why (A5) is correct also for $j'_1 = 1$ or $j'_{k-2} = N$. One of the sums over $\boldsymbol{j}_{1,\ell}$ and $\boldsymbol{j}_{k-1,\ell}$ in (A4) is then empty and evaluates to zero, corresponding to the fact that $\boldsymbol{j}_{1,\ell} \in N(k)$ or $\boldsymbol{j}_{k-1,\ell} \in N(k)$ cannot be achieved for any ℓ. In (A5), on the other hand, the corresponding terms drop out because $\boldsymbol{j}_{1,\ell}$ contains index pairs.

The evaluation of (A5) requires us to keep track of the first two components of $\boldsymbol{j}_{1,\ell}$, namely $(\ell, \ell+1)$, in the Green function (30). To do so we focus on the subset of permutations $\mathcal{S}_{\mu,\nu}(k) \subset \mathcal{S}(k)$ that map these first two components onto components μ, ν, respectively. Any

$\pi \in \mathcal{S}_{\mu,\nu}(k)$ may be factorized into $\pi = \pi_{\mu,\nu} \circ \bar{\pi}$ where $\bar{\pi}$ performs an arbitrary permutation on the components $3, \ldots, k$ and $\pi_{\mu,\nu}$ then puts the components $1, 2$ into positions μ, ν, respectively. We have $(-1)^{\pi} = (-1)^{\pi_{\mu,\nu}}(-1)^{\bar{\pi}}$, where $(-1)^{\pi_{\mu,\nu}} = (-1)^{\nu-\mu-1}$ for $\nu > \mu$ and $(-1)^{\pi_{\mu,\nu}} = -(-1)^{\mu-\nu-1}$ for $\nu < \mu$ follows by counting the number of transpositions in $\pi_{\mu,\nu}$. Using this factorization for $\pi \in \mathcal{S}_{\mu,\nu}(k)$ and the fact $\bigcup_{\mu\neq\nu} \mathcal{S}_{\mu,\nu}(k) = \mathcal{S}(k)$ allows us to rewrite the Green function (30) in the form

$$G^{(k)}_{\boldsymbol{i},\boldsymbol{j}_{1,\ell}}(t) = \sum_{1\leqslant\mu<\nu\leqslant k} (-1)^{\nu-\mu-1}\, \mathrm{e}^{-2t} I_{i_\mu - l}(\gamma t) I_{i_\nu - l - 1}(\gamma t) G^{(k-2)}_{\boldsymbol{i}\setminus(i_\mu,i_\nu),\boldsymbol{j}'}(t) - \sum_{1\leqslant\nu<\mu\leqslant k} (-1)^{\mu-\nu-1}\, \mathrm{e}^{-2t} I_{i_\mu - l}(\gamma t) I_{i_\nu - l - 1}(\gamma t) G^{(k-2)}_{\boldsymbol{i}\setminus(i_\nu,i_\mu),\boldsymbol{j}'}(t). \tag{A6}$$

Relabelling $\mu \leftrightarrow \nu$ in the second line of (A6) and substitution into (A5) then gives

$$\sum_{\boldsymbol{j}\in N(k)} G^{(k)}_{\boldsymbol{i},\boldsymbol{j}}(t) b^{(k)}_{\boldsymbol{j},\boldsymbol{j}'} = \gamma \sum_{1\leqslant\mu<\nu\leqslant k} (-1)^{\nu-\mu-1}\, \mathrm{e}^{-2t} \times \sum_{\ell=1}^{N} \left[I_{i_\mu - l} I_{i_\nu - l - 1} - I_{i_\nu - l} I_{i_\mu - l - 1} \right](\gamma t) G^{(k-2)}_{\boldsymbol{i}\setminus(i_\mu,i_\nu),\boldsymbol{j}'}(t). \tag{A7}$$

Our derivation of the identity (A1) is completed by using the convolution property (B8) of the functions $I_n(x)$ to evaluate the ℓ-sum

$$\gamma\, \mathrm{e}^{-2t} \sum_{\ell=1}^{N} \left[I_{i_\mu - l} I_{i_\nu - l - 1} - I_{i_\nu - l} I_{i_\mu - l - 1} \right](\gamma t) = \gamma\, \mathrm{e}^{-2t} \left[I_{i_\nu - i_\mu - 1} - I_{i_\nu - i_\mu + 1} \right](2\gamma t) \tag{A8}$$

and identifying (A8) as the t-derivative of $H_{i_\nu - i_\mu}(2t)$ given by (C1).

Appendix B. Properties of $I_n(x)$

The functions $I_n(x)$ appear in our analysis of the Glauber–Ising model when we solve (27) for $\bar{\Phi}$ and invert the Fourier transforms. We define

$$I_n(x) = \frac{1}{N} \sum_{q\in Q} \mathrm{e}^{\mathrm{i}nq + x\cos q} = \frac{1}{N} \sum_{q\in Q} \cos(nq)\, \mathrm{e}^{x\cos q} \tag{B1}$$

where we have to set $Q = Q_{\mathrm{e}}$ on even levels k of the hierarchy and $Q = Q_{\mathrm{o}}$ on odd ones. In the thermodynamic limit $N \to \infty$ the sets Q_{e}, Q_{o} given in (25), (26) become dense on $[0, 2\pi]$. Hence for any fixed $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ we have

$$\lim_{N\to\infty} I_n(x) = \int_0^{2\pi} \frac{\mathrm{d}q}{2\pi} \cos(nq)\, \mathrm{e}^{x\cos q} = \mathrm{I}_n(x) \tag{B2}$$

that is the $I_n(x)$ on all levels k become modified Bessel functions $\mathrm{I}_n(x)$ as $N \to \infty$. For a comprehensive discussion of the $\mathrm{I}_n(x)$ see [26]. Many properties of the $\mathrm{I}_n(x)$ in fact have an analogue for finite N. The following identities hold for $Q = Q_{\mathrm{e}}$ as well as $Q = Q_{\mathrm{o}}$:

$$\forall 0 \leqslant n < N : I_n(0) = \delta_{n,0} \tag{B3}$$

$$\forall x \in \mathbb{R} \quad \forall n \in \mathbb{Z} : I_{-n}(x) = I_n(x) \tag{B4}$$

$$\forall x \in \mathbb{R}^+ \quad \forall -\left\lfloor \frac{N}{2} \right\rfloor \leqslant n \leqslant \left\lfloor \frac{N}{2} \right\rfloor : I_n(x) \geqslant 0 \tag{B5}$$

$$\forall x \in \mathbb{R} \quad \forall n \in \mathbb{Z} : \frac{\mathrm{d}}{\mathrm{d}x} I_n(x) = \frac{1}{2} \left[I_{n-1}(x) + I_{n+1}(x) \right] \tag{B6}$$

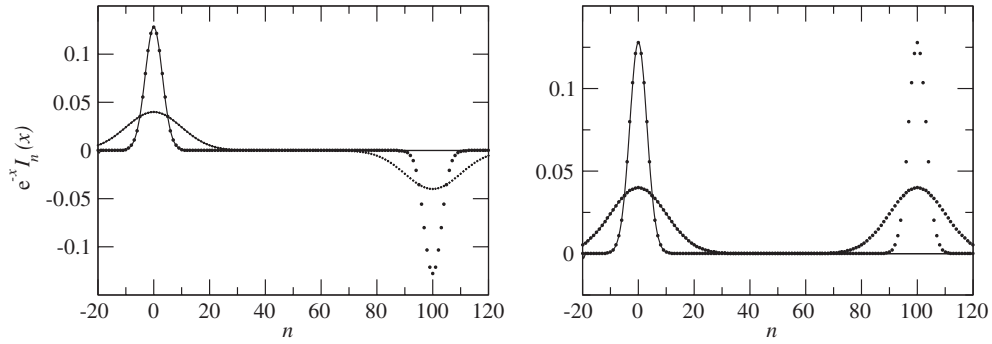

Figure 4. Plots of the scaled functions $e^{-x} I_n(x)$ for a finite system of size $N = 100$ (dots) and the $N \to \infty$ limit $e^{-x} I_n(x)$ (lines). The (left) right figure shows the N-(anti) periodic functions associated with (even) odd levels, respectively. The curves correspond to $x = 10$ (narrow peaks) and $x = 100$ (broader peaks).

$$\forall x \in \mathbb{R} \quad \forall m, n \in \mathbb{Z} \quad \forall q \in Q_{\mathrm{e/o}} : \sum_{k=m+1}^{m+N} e^{ikq} I_{k+n}^{\mathrm{e/o}}(x) = e^{-inq + x\cos q} \tag{B7}$$

$$\forall x, y \in \mathbb{R} \quad \forall m, n \in \mathbb{Z} : \sum_{k=m+1}^{m+N} I_k^{\mathrm{e/o}}(x) I_{n-k}^{\mathrm{e/o}}(y) = I_n^{\mathrm{e/o}}(x+y). \tag{B8}$$

In (B7), (B8) the e/o symbols indicate that the identities hold on even as well as odd levels but not between even and odd ones. The functions $I_n(x)$ associated with even and odd levels differ in the following properties (here $m, n \in \mathbb{Z}$ and $x \in \mathbb{R}$):

$$\begin{array}{ll} \text{odd levels}(Q = Q_{\mathrm{o}}) & \text{even levels}(Q = Q_{\mathrm{e}}) \\ I_{n+N}(x) = I_n(x) & I_{n+N}(x) = -I_n(x) \end{array} \tag{B9}$$

$$\sum_{k=m+1}^{m+N} I_k(x) = e^x \qquad \sum_{k=m+1}^{m+N} |I_k(x)| \leqslant e^x. \tag{B10}$$

The identities (B3), (B4), (B6) follow immediately from the definition of $I_n(x)$. (B7)–(B9), on the other hand, are a consequence of the convolution property and N-(anti) periodicity of the discrete Fourier transforms (23), (24). We have not managed to prove the seemingly trivial inequality (B5) in a rigorous way. Numerical evaluations of (B1), however, leave no doubt that (B5) should be true in general. Assuming that (B5) is true, (B10) can be verified as follows: on odd levels, where $I_n(x)$ is N-periodic, (B5) implies that $|I_n(x)| = I_n(x)$ for all $n \in \mathbb{Z}$ and hence (B10) follows from (B7) by setting $q = 0$; this is allowed since $0 \in Q_{\mathrm{o}}$. On even levels, the N-antiperiodicity of the $I_n(x)$ is cancelled by the modulus in (B10). Hence we may shift the summation range to the region $-\lfloor \frac{N-1}{2} \rfloor \leqslant n \leqslant \lfloor \frac{N}{2} \rfloor$ where $I_n(x)$ is positive and drop the modulus. Then, by substituting (B1) and exchanging sums, the bound (B10) can be verified [16]. To illustrate the actual shape of the $I_n(x)$ we show some plots in figure 4.

Appendix C. Properties of $H_n(t)$ and related functions

The functions $H_n(t)$ originate from the projection of the Green function (A1), which is an essential ingredient for our solution (40). Appendix A motivates the definition

$$H_n(t) = \frac{\gamma}{2}\int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\tau}\left[I_{n-1}(\gamma\tau) - I_{n+1}(\gamma\tau)\right]. \tag{C1}$$

When substituting (B1) for $I_n(x)$ the integral in (C1) can be solved and we obtain the alternative representation

$$H_n(t) = \frac{1}{N}\sum_{q\in Q}\sin(nq)\frac{\gamma\sin q}{1-\gamma\cos q}[1-\mathrm{e}^{-t(1-\gamma\cos q)}]. \tag{C2}$$

In (C2) one has to substitute $Q = Q_{\mathrm{e/o}}$ as appropriate. Like the $I_n(x)$, the $H_n(x)$ thus have slightly different properties at even and odd levels of the hierarchy.

Two obvious properties of $H_n(t)$ are $H_{-n}(t) = -H_n(t)$ and $H_n(0) = 0$. In the thermodynamic limit $N\to\infty$ uniform convergence of $I_n(x)$ on any finite interval $x\in[0,a]$ and for fixed $n\in\mathbb{Z}$ allows us to replace $I_n(x)$ in (C1) by its limit $\mathrm{I}_n(x)$. In (C2), on the other hand, the limit $N\to\infty$ amounts to replacing $\frac{1}{N}\sum_q$ by $\frac{1}{2\pi}\int \mathrm{d}q$. In analogy with appendix B we use the notation $\mathrm{H}_n(t)$ for the $N\to\infty$ limit of $H_n(t)$. Further properties of the functions $H_n(t)$ may be obtained directly from (C1) using the features of $I_n(x)$ given in appendix B. It is interesting to note that, for $\gamma = 1$, $\mathrm{H}_n(2t)$ is essentially the probability that two independent, continuous time random walkers on a one-dimensional lattice, which are a distance $n > 0$ apart at time $t = 0$, meet before time t [15].

In the context of sections 5, 6 the following sum, which we use to extend the definition of H_n to two time arguments, is of relevance

$$H^{\mathrm{e/o}}_{j-i}(t_1,t_2) = \sum_n \mathrm{e}^{-t_1} I^{\mathrm{e/o}}_{i-n}(\gamma t_1) H^{\mathrm{e/o}}_{j-n}(t_2). \tag{C3}$$

Here e/o emphasizes that both I and H on the rhs of (C3) must be associated either with even or odd levels. From (B8) and (C1) we immediately obtain

$$H_n(t_1,t_2) = \frac{\gamma}{2}\int_{t_1}^{t_1+t_2}\mathrm{d}\tau\, \mathrm{e}^{-\tau}\left[I_{n-1}(\gamma\tau) - I_{n+1}(\gamma\tau)\right] \tag{C4}$$

and thence, or from (B7) and (C2),

$$H_n(t_1,t_2) = \frac{1}{N}\sum_{q\in Q}\sin(nq)\frac{\gamma\sin q}{1-\gamma\cos q}\,\mathrm{e}^{-t_1(1-\gamma\cos q)}[1-\mathrm{e}^{-t_2(1-\gamma\cos q)}]. \tag{C5}$$

According to e.g. (C1), (C4) we have the obvious link $H_n(t) = H_n(0,t)$ between functions with one and two time arguments. But it is quite remarkable that we may conversely write

$$H_n(t_1,t_2) = H_n(t_1+t_2) - H_n(t_1) \tag{C6}$$

when considering the definition (C3). Expressions (C4), (C5) are particularly useful in the thermodynamic limit $N\to\infty$ where (C3) becomes an infinite summation. In (C4), (C5), however, this limit is straightforward. As for (C1), (C2) we then use the symbol $\mathrm{H}_n(t_1,t_2)$.

In section 4 we obtained another function $H''_n(t)$ as given by (52) that is closely related to $H_n(t)$. The physical relevance of $H''_n(t)$ is discussed in the text below (52). In mathematical terms we may simply consider $H''_n(t)$ as a generalization of $H_n(t)$ since we have $H''_n(t) = H_n(t)$ for $\gamma_{\mathrm{i}} = 0$. For $\gamma_{\mathrm{i}}\neq 0$, a representation of $H''_n(t)$ in the form (C1) is not possible. Note that while $H_n(t)$ is defined on even as well as odd levels of the hierarchy, $H''_n(t)$ is always associated with even levels, i.e. the sum in (52) runs over Q_{e}. In analogy with (C3) we now extend the definition of $H''_n(t)$ to two time arguments. The following sums are relevant for appendix D

$$H''_{j-i}(t_1,t_2) = \sum_n \mathrm{e}^{-t_1} I^{\mathrm{e}}_{i-n}(\gamma t_1) H''_{j-n}(t_2) \tag{C7}$$

$$\tilde{H}''_{j-i}(t_1,t_2)=\sum_n \mathrm{sgn}(j-n)\,\mathrm{e}^{-t_1}I^{\mathrm{o}}_{i-n}(\gamma t_1)H''_{j-n}(t_2) \tag{C8}$$

where $1\leqslant i,\,j\leqslant N$ and the sum runs over $n=1,\ldots,N$ for a finite system. The superscripts e and o of $I_n(x)$ indicate that these functions must be associated with even and odd levels of the hierarchy, respectively. The sum (C7) may be evaluated easily using (52) and (B7)

$$H''_n(t_1,t_2)=\frac{1}{N}\sum_{q\in Q_{\mathrm{e}}}\sin(nq)\,\mathrm{e}^{-t_1(1-\gamma\cos q)} \times\left\{\frac{\gamma_{\mathrm{i}}\sin q}{1-\gamma_{\mathrm{i}}\cos q}\mathrm{e}^{-t_2(1-\gamma\cos q)}+\frac{\gamma\sin q}{1-\gamma\cos q}[1-\mathrm{e}^{-t_2(1-\gamma\cos q)}]\right\}. \tag{C9}$$

In analogy with H we obtain from (52), (C2) and (C9) the link $H''_n(t)=H''_n(0,t)$ and the non-trivial identity

$$H''_n(t_1,t_2)=H''_n(t_1+t_2)-H_n(t_1) \tag{C10}$$

for the sum (C7). For $\gamma_{\mathrm{i}}=0$ equations (52), (C7), (C9) and (C10) reduce to (C2), (C3), (C5) and (C6), respectively, with H'' replaced by H. Consistent with our notation elsewhere we will denote the $N\to\infty$ limit of H'' by H''.

In contrast to (C7), the sum (C8) cannot be evaluated explicitly. But it may be rewritten in terms of the integral representation

$$\tilde{H}''_n(t_1,t_2)=\mathrm{sgn}(n)H''_n(t_1,t_2)+\gamma\int_0^{t_1}\mathrm{d}\tau\,\mathrm{e}^{-\tau}I^{\mathrm{o}}_n(\gamma\tau)H''_1(t_1-\tau,t_2). \tag{C11}$$

Note that, in contrast to H''_n, the function $\tilde{H}''_n$ is even in n. As for (C7), (C9) we will drop the double-primes in (C8), (C11) for $\gamma_{\mathrm{i}}=0$, and denote the $N\to\infty$ limit of $\tilde{H}''_n(t_1,t_2)$ by $\tilde{\mathrm{H}}''_n(t_1,t_2)$. Due to uniform convergence of the integrand, this limit is obtained by replacing I, H with I, H in (C11).

Equivalence of (C8) and (C11) may be verified as follows: substitute the expression (52) for $H''_n(t)$ into (C8) and exchange the sums over n and q. Now, the sum over n is of the form

$$f_{i,j}(x,q)=\sum_{n=1}^{N}\mathrm{sgn}(j-n)I^{\mathrm{o}}_{i-n}(x)\sin\left[q(j-n)\right] \tag{C12}$$

where q is the summation variable in H'' and thus $q\in Q_{\mathrm{e}}$. Differentiating (C12) w.r.t. x using (B6), shifting the summation variable $n\to n\pm1$ as appropriate and rewriting the result in terms of sums over the range $n=1,\ldots,N$ yields

$$\begin{aligned}\frac{\partial}{\partial x}f_{i,j}(x,q)=&\frac{1}{2}\sum_{n=1}^{N}\mathrm{sgn}(j-n+1)I^{\mathrm{o}}_{i-n}(x)\sin[q(j-n+1)]\\&+\frac{1}{2}\mathrm{sgn}(j-N)I^{\mathrm{o}}_{i-N-1}(x)\sin[q(j-N)]-\frac{1}{2}\mathrm{sgn}(j)I^{\mathrm{o}}_{i-1}(x)\sin(qj)\\&+\frac{1}{2}\sum_{n=1}^{N}\mathrm{sgn}(j-n-1)I^{\mathrm{o}}_{i-n}(x)\sin[q(j-n-1)]\\&+\frac{1}{2}\mathrm{sgn}(j-1)I^{\mathrm{o}}_{i}(x)\sin[q(j-1)]\\&-\frac{1}{2}\mathrm{sgn}(j-N-1)I^{\mathrm{o}}_{i-N}(x)\sin[q(j-N-1)].\end{aligned} \tag{C13}$$

Next focus on the second line in (C13). Due to N-periodicity (B9) of the functions $I_n(x)$ on odd levels we have $I^{\mathrm{o}}_{i-N-1}(x)=I^{\mathrm{o}}_{i-1}(x)$. Also, since $q\in Q_{\mathrm{e}}$ as given by (25), the identity

$\sin[q(j-N)] = -\sin(qj)$ holds. Recalling, finally, that $1 \leqslant j \leqslant N$ is assumed and noting that $\mathrm{sgn}(j-N) = -\mathrm{sgn}(j)$ for $1 \leqslant j < N$ thus shows that the two terms in the second line of (C13) cancel each other for $1 \leqslant j < N$. When $j = N$ both terms are zero because $\sin(0) = \sin(Nq) = 0$. It follows similarly that the fourth line in (C13) drops out. Expressing $\mathrm{sgn}(j-n\pm1)$ via $\mathrm{sgn}(j-n)$ by adding compensating terms for $j = n, n\pm1$ and using the trigonometric identity $\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$ allows us to combine the sums in (C13) in the form (C12). Consequently the $f_{i,j}(x,q)$ satisfy the inhomogeneous differential equations

$$\frac{\partial}{\partial x} f_{i,j}(x,q) - \cos(q) f_{i,j}(x,q) = I^{\mathrm{o}}_{i-j}(x)\sin(q). \tag{C14}$$

From (B3) and (C12) we also have $f_{i,j}(0,q) = \mathrm{sgn}(j-i)\sin[q(j-i)]$. Thus we obtain $f_{i,j}(x,q)$ by integrating (C14) for this initial condition, i.e.

$$f_{i,j}(x,q) = \mathrm{sgn}(j-i)\sin[q(j-i)]\,\mathrm{e}^{x\cos q} + \sin q \int_0^x \mathrm{d}x'\,\mathrm{e}^{(x-x')\cos q} I^{\mathrm{o}}_{i-j}(x'). \tag{C15}$$

Using (C15) in (C8) immediately produces (C11) if one bears in mind (C9).

We conclude this section by showing that the general expressions (C9), (C11) simplify quite significantly if one considers a quench from the random initial configuration corresponding to $T_{\mathrm{i}} = \infty$ to $T = 0$, i.e. $\gamma_{\mathrm{i}} = 0$ and $\gamma = 1$. Let us first focus on $H''_n(t_1,t_2)$ as given by (C9), which for $\gamma_{\mathrm{i}} = 0$ we write as $H_n(t_1,t_2)$. Since also $\gamma = 1$, we may use the trigonometric identity

$$\sin[(n+1)x]\frac{\sin x}{1-\cos x} = \sin(nx)\frac{\sin x}{1-\cos x} + \cos(nx) + \cos[(n+1)x] \tag{C16}$$

in (C5). This yields the zero temperature recursion formula

$$H_{n+1}(t_1,t_2) = H_n(t_1,t_2) + \mathrm{e}^{-t_1}\left[I_n + I_{n+1}\right](t_1) - \mathrm{e}^{-(t_1+t_2)}\left[I_n + I_{n+1}\right](t_1+t_2) \tag{C17}$$

for $H_n(t_1,t_2)$. Starting from $H_0(t_1,t_2) = 0$, any $H_n(t_1,t_2)$ can thus be expressed purely in terms of functions $I_n(x)$. By setting $t_1 = 0$ in (C17) and using (B3) we also obtain a recursion for $H_n(t) = H_n(0,t)$, that is

$$H_1(t) = 1 - \mathrm{e}^{-t}\left[I_0 + I_1\right](t) \qquad \text{and} \qquad H_{n+1}(t) = H_n(t) - \mathrm{e}^{-t}\left[I_n + I_{n+1}\right](t) \tag{C18}$$

where $1 < n+1 < N$. Outside that range we have $H_{-n}(t) = -H_n(t)$ and N-(anti)-periodicity in n according to e.g. (C1). Equations (C17), (C18) apply in the thermodynamic limit, too, if we replace H by H and $I_n(x)$ by modified Bessel functions $\mathrm{I}_n(x)$.

Now we consider $\tilde{H}''_n(t_1,t_2)$ at $\gamma_{\mathrm{i}} = 0$, $\gamma = 1$ and, for the sake of simplicity, also $N \to \infty$. As explained above, we drop the double-primes in (C11) for $\gamma_{\mathrm{i}} = 0$ and replace H, I by H, I in the thermodynamic limit. This gives

$$\tilde{\mathrm{H}}_n(t_1,t_2) = \mathrm{sgn}(n)\mathrm{H}_n(t_1,t_2) + \int_0^{t_1} \mathrm{d}\tau\,\mathrm{e}^{-\tau}\mathrm{I}_n(\tau)\mathrm{H}_1(t_1-\tau,t_2). \tag{C19}$$

Expressing $\mathrm{H}_1(t_1-\tau,t_2)$ via (C17) allows us to rewrite (C19) as

$$\begin{aligned}\tilde{\mathrm{H}}_n(t_1,t_2) = {}& \mathrm{sgn}(n)\mathrm{H}_n(t_1,t_2) + \mathrm{e}^{-t_1} f_n(t_1) - \mathrm{e}^{-(t_1+t_2)} f_n(t_1+t_2) \\ & + \mathrm{e}^{-(t_1+t_2)} \int_0^{t_2} \mathrm{d}\tau\,\mathrm{I}_n(t_1+t_2-\tau)\left[\mathrm{I}_0 + \mathrm{I}_1\right](\tau)\end{aligned} \tag{C20}$$

where the functions $f_n(t)$ are given by the single-sided convolution integral

$$f_n(t) = \int_0^t \mathrm{d}\tau\,\mathrm{I}_n(\tau)\left[\mathrm{I}_0 + \mathrm{I}_1\right](t-\tau). \tag{C21}$$

By Laplace transforming (C21) it is easy to verify that $f_0(t) = \mathrm{e}^t - \mathrm{I}_0(t)$ and $f_{n+1}(t) = f_n(t) - 2\mathrm{I}_{n+1}(t)$. This recursion formula produces a very similar expression for $\mathrm{e}^{-t} f_n(t)$ as (C18) for $H_n(t)$ which in fact amounts to the link

$$\mathrm{e}^{-t}\mathrm{I}_n(t) + \mathrm{e}^{-t} f_n(t) = \delta_{n,0} + \mathrm{sgn}(n)\mathrm{H}_n(t). \tag{C22}$$

Substituting (C22) into (C20) and using (C6) reduces the expression (C19) for $\tilde{\mathrm{H}}_n(t_1, t_2)$ to

$$\mathrm{e}^{-t_1}\mathrm{I}_n(t_1) + \tilde{\mathrm{H}}_n(t_1, t_2) = \mathrm{e}^{-(t_1+t_2)} \left\{ \mathrm{I}_n(t_1 + t_2) + \int_0^{t_2} \mathrm{d}\tau\, \mathrm{I}_n(t_1 + t_2 - \tau)\, [\mathrm{I}_0 + \mathrm{I}_1]\,(\tau) \right\}. \tag{C23}$$

Appendix D. The sums $\mathcal{E}$ and $\mathcal{F}$

Here we focus on the sums $\mathcal{E}, \mathcal{F}$ given in (61), (62). These appear in the expressions for the homogeneous solutions of the two-time correlation and response functions discussed in sections 5, 6. Our intention in this section is to reduce the sums in (61), (62) over ranges scaling with N to sums over a finite number of terms, which therefore remain manageable even for $N \to \infty$.

Let us first consider the one-dimensional sum $\mathcal{E}$ given in (61), where we use the abbreviation $p = \dim(\boldsymbol{j})$. For a finite system we have $\boldsymbol{j} \in N(p)$ and the summation range is $m = 1, 2, \ldots, N$. If p is even, then the product over the sgn's in (61) is +1 for $m < j_1$ and $m > j_p$. Therefore the expression

$$\boldsymbol{\delta}_{\boldsymbol{j},m} = 1 - \prod_{\lambda=1}^{\dim(\boldsymbol{j})} \mathrm{sgn}(j_\lambda - m) \tag{D1}$$

can be non-zero only for $j_1 \leqslant m \leqslant j_p$. In fact we have $\boldsymbol{\delta}_{\boldsymbol{j},j_\nu} = 1$ for $\nu = 1, \ldots, p$ and $\boldsymbol{\delta}_{\boldsymbol{j},m} = 2$ for $j_{2\nu-1} < m < j_{2\nu}$ with $\nu = 1, \ldots, p/2$. Otherwise $\boldsymbol{\delta}_{\boldsymbol{j},m}$ is zero. In terms of (D1) we may rewrite (61) as

$$\mathcal{E}^{\boldsymbol{j}}_{i_\varepsilon, j_\nu}(t, t') = \sum_m [1 - \boldsymbol{\delta}_{\boldsymbol{j},m}]\, \mathrm{e}^{-\Delta} I_{i_\varepsilon - m}(\gamma\Delta) H''_{j_\nu - m}(2t'). \tag{D2}$$

Now recall that $C^{(k,l)}(t, t')$ and $R^{(k,l)}(t, t')$ are non-vanishing only if $k + l$ is even. Hence if $p = l$ (or $p = l \pm 2$ for responses where from (68) the argument $\boldsymbol{j}$ in (D2) might be a vector $\boldsymbol{j}^{\nu,\mathrm{s}}$) is even, then so must k be. Consequently the functions $I_n(x)$ in (D2) are associated with even levels. This allows us to rewrite (D2)

$$\mathcal{E}^{\boldsymbol{j}}_{i_\varepsilon, j_\nu}(t, t') = H''_{j_\nu - i_\varepsilon}(\Delta, 2t') - \sum_m \boldsymbol{\delta}_{\boldsymbol{j},m} \mathrm{e}^{-\Delta} I_{i_\varepsilon - m}(\gamma\Delta) H''_{j_\nu - m}(2t') \tag{D3}$$

using (C7). The remaining summation in (D3) only produces non-zero terms in the N-independent range $j_1 \leqslant m \leqslant j_p$. Consequently, the thermodynamic limit $N \to \infty$ of (D3) is straightforwar.

Following a similar strategy we now rewrite the sum $\mathcal{E}$ for $p = \dim(\boldsymbol{j})$ odd. In this case the product over the sgn's in (61) gives +1 for $m < j_1$ but -1 for $m > j_p$. Hence we introduce

$$\boldsymbol{\sigma}^a_{\boldsymbol{j},m} = \mathrm{sgn}(a - m) - \prod_{\lambda=1}^{\dim(\boldsymbol{j})} \mathrm{sgn}(j_\lambda - m) \tag{D4}$$

where a is some reference site that should be chosen in the range $j_1 \leqslant a \leqslant j_p$. Then (D4) again only produces non-zero values for $j_1 \leqslant m \leqslant j_p$. In terms of (D4) and with the choice $a = j_\nu$ the sum (61) becomes

$$\mathcal{E}^{\boldsymbol{j}}_{i_\varepsilon, j_\nu}(t, t') = \sum_m \left[\mathrm{sgn}(j_\nu - m) - \boldsymbol{\sigma}^{j_\nu}_{\boldsymbol{j},m}\right] \mathrm{e}^{-\Delta} I_{i_\varepsilon - m}(\gamma\Delta) H''_{j_\nu - m}(2t'). \tag{D5}$$

Applying the same reasoning as below (D2) we conclude that in (D5) the functions $I_n(x)$ must be associated with odd levels. Hence we utilize (C8) to rewrite (D5) as

$$\mathcal{E}^{\boldsymbol{j}}_{i_\varepsilon,j_\nu}(t,t') = \tilde{H}''_{j_\nu - i_\varepsilon}(\Delta, 2t') - \sum_m \boldsymbol{\sigma}^{j_\nu}_{\boldsymbol{j},m}\, \mathrm{e}^{-\Delta} I_{i_\varepsilon - m}(\gamma\Delta) H''_{j_\nu - m}(2t'). \tag{D6}$$

Because the summation range in (D6) that produces non-zero terms is again N-independent and bounded by $j_1 \leqslant m \leqslant j_p$, the thermodynamic limit of (D6) is obvious.

For the two-dimensional sum $\mathcal{F}$ given in (62) with even $p = \dim \boldsymbol{j}$ we may proceed in full analogy with $\mathcal{E}$. Expressing the products over the sgn's in (62) by (D1) yields

$$\mathcal{F}^{\boldsymbol{j}}_{i_\varepsilon,i_\delta}(t,t') = \sum_{m,n}[1 - \boldsymbol{\delta}_{\boldsymbol{j},m} - \boldsymbol{\delta}_{\boldsymbol{j},n} + \boldsymbol{\delta}_{\boldsymbol{j},m}\boldsymbol{\delta}_{\boldsymbol{j},n}]\mathrm{e}^{-2\Delta} I_{i_\varepsilon - m}(\gamma\Delta) I_{i_\delta - n}(\gamma\Delta) H''_{n-m}(2t'). \tag{D7}$$

The unrestricted one- and two-dimensional sums in (D7) may be evaluated using (B8) and (C7). We are then left with bounded summation ranges $j_1 \leqslant m, n \leqslant j_p$

$$\begin{aligned}\mathcal{F}^{\boldsymbol{j}}_{i_\varepsilon,i_\delta}(t,t') = {} & H''_{i_\delta - i_\varepsilon}(2\Delta, 2t') \\ & - \sum_m \boldsymbol{\delta}_{\boldsymbol{j},m} \mathrm{e}^{-\Delta}\left[I_{i_\varepsilon - m}(\gamma\Delta) H''_{i_\delta - m}(\Delta, 2t') - I_{i_\delta - m}(\gamma\Delta) H''_{i_\varepsilon - m}(\Delta, 2t')\right] \\ & + \sum_{m,n} \boldsymbol{\delta}_{\boldsymbol{j},m}\boldsymbol{\delta}_{\boldsymbol{j},n} \mathrm{e}^{-2\Delta} I_{i_\varepsilon - m}(\gamma\Delta) I_{i_\delta - n}(\gamma\Delta) H''_{n-m}(2t').\end{aligned} \tag{D8}$$

Rewriting the two-dimensional sum $\mathcal{F}$ for $p = \dim(\boldsymbol{j})$ odd is somewhat problematic. If we express the products over the sgn's in (62) using (D4) it is not possible to choose reference points a s.t. we arrive at sums of the form (C8). It turns out that the most compact—yet still bulky—result follows when choosing

$$\begin{aligned}\mathcal{F}^{\boldsymbol{j}}_{i_\varepsilon,i_\delta}(t,t') = {} & \sum_{m,n}\left[\mathrm{sgn}(i_\varepsilon - m)\mathrm{sgn}(i_\delta - n) - \mathrm{sgn}(i_\varepsilon - m)\boldsymbol{\sigma}^{i_\delta}_{\boldsymbol{j},n} - \mathrm{sgn}(i_\delta - n)\boldsymbol{\sigma}^{i_\varepsilon}_{\boldsymbol{j},m} + \boldsymbol{\sigma}^{i_\varepsilon}_{\boldsymbol{j},m}\boldsymbol{\sigma}^{i_\delta}_{\boldsymbol{j},n}\right] \\ & \times \mathrm{e}^{-2\Delta} I_{i_\varepsilon - m}(\gamma\Delta) I_{i_\delta - n}(\gamma\Delta) H''_{n-m}(2t').\end{aligned} \tag{D9}$$

By generalizing (C12) integral representations for the sums in (D9) may be derive. One should find

$$\begin{aligned}\mathcal{F}^{\boldsymbol{j}}_{i_\varepsilon,i_\delta}(t,t') = {} & \sum_{m,n} \boldsymbol{\sigma}^{i_\varepsilon}_{\boldsymbol{j},m}\boldsymbol{\sigma}^{i_\delta}_{\boldsymbol{j},n} \mathrm{e}^{-2\Delta} I_{i_\varepsilon - m}(\gamma\Delta) I_{i_\delta - n}(\gamma\Delta) H''_{n-m}(2t') \\ & - \frac{\gamma}{2}\sum_m \boldsymbol{\sigma}^{i_\varepsilon}_{\boldsymbol{j},m}\, \mathrm{e}^{-\Delta} I_{i_\varepsilon - m}(\gamma\Delta) \int_0^\Delta \mathrm{d}\tau\, \mathrm{e}^{-\tau} I_0(\gamma\tau) \\ & \times [H''_{i_\delta - m - 1} - H''_{i_\delta - m + 1}](\Delta - \tau, 2t') \\ & + \frac{\gamma}{2}\sum_m \boldsymbol{\sigma}^{i_\delta}_{\boldsymbol{j},m}\, \mathrm{e}^{-\Delta} I_{i_\delta - m}(\gamma\Delta) \int_0^\Delta \mathrm{d}\tau\, \mathrm{e}^{-\tau} I_0(\gamma\tau) \\ & \times [H''_{i_\varepsilon - m - 1} - H''_{i_\varepsilon - m + 1}](\Delta - \tau, 2t') \\ & + \frac{\gamma^2}{4}\int_0^\Delta \mathrm{d}\tau_1\, \mathrm{e}^{-\tau_1} I_0(\gamma\tau_1) \int_0^\Delta \mathrm{d}\tau_2\, \mathrm{e}^{-\tau_2} I_0(\gamma\tau_2) \\ & \times [-H''_{i_\delta - i_\varepsilon - 2} + 2H''_{i_\delta - i_\varepsilon} - H''_{i_\delta - i_\varepsilon + 2}](2\Delta - \tau_1 - \tau_2, 2t').\end{aligned} \tag{D10}$$

As the reference points in (D9) are i_ε and i_δ the bounds on the summation ranges in (D10) are of the form $\min(i_\varepsilon, j_1) \leqslant m \leqslant \max(i_\varepsilon, j_l)$, etc. For $\gamma_1 = 0$ and $\gamma = 1$ the expression (D10) may be simplified using (C17).

Appendix E. The notation of equation (71)

In this appendix we illustrate by means of two simple examples how arbitrary homogeneous solutions for two-time multispin response functions may be obtained from (71).

Let us first consider the simplest possible case of the spin response function to a local magnetic fiel. Here we have $k = l = 1$ and $\boldsymbol{i} = (i)$, $\boldsymbol{j} = (j)$. According to the formalism presented in section 6 the first step in the evaluation of (71) is to determine the squashed index vectors (68). In the case at hand there is only one such vector since $l = 1$, namely $\boldsymbol{j}^{1,\mathrm{s}} = \boldsymbol{j}^1 = (j-1, j, j+1)$ with $l^{1,\mathrm{s}} = 3$. Then, by setting $k = l = 1$ in (71) and substituting $l^{1,\mathrm{s}}$, we obtain

$$\Phi_{i,j}^{(1,1)}(t,t') = \mathrm{e}^{-\Delta} I_{i-j}(\gamma\Delta)\left[\left(1-\frac{\gamma^2}{2}\right)\sum_{\pi\in\mathcal{P}'(1)}(-1)^{\pi}\prod_{\lambda=1}^{1}\mathcal{V}'^{j}_{(i\cup j)_{\pi(2\lambda-1)},(i\cup j)_{\pi(2\lambda)}}(t,t') + \frac{\gamma^2}{2}\sum_{\pi\in\mathcal{P}'(2)}(-1)^{\pi}\prod_{\lambda=1}^{2}\mathcal{V}'^{\boldsymbol{j}^{1,\mathrm{s}}}_{(i\cup\boldsymbol{j}^{1,\mathrm{s}})_{\pi(2\lambda-1)},(i\cup\boldsymbol{j}^{1,\mathrm{s}})_{\pi(2\lambda)}}(t,t')\right].$$

The summations over pairings in this equation are restricted to those, $\mathcal{P}'$, that contain the pair (i, j), where we set $\mathcal{V}'_{i,j} = 1$. In the first sum we have $\mathcal{P}(1) = \{\mathrm{Id}\}$ and pairs are drawn from $i \cup j = (i, j)$. Hence $\mathcal{P}'(1) = \{\mathrm{Id}\}$ and the only term in this sum is $\mathcal{V}'_{i,j} = 1$. In the second sum, however, we have to consider the three pairings $(1, 2, 3, 4)$, $(1, 3, 2, 4)$ and $(1, 4, 2, 3)$ contained in $\mathcal{P}(2)$. As pairs are drawn from $i \cup \boldsymbol{j}^{1,\mathrm{s}} = (i, j-1, j, j+1)$ the only pairing contained in $\mathcal{P}'(2)$ is $(1, 3, 2, 4)$; the sign of the corresponding permutation π is $(-1)^{\pi} = -1$. So there is again only one term in this sum, namely $-\mathcal{V}'_{i,j}\mathcal{V}'_{j-1,j+1} = -\mathcal{V}_{j-1,j+1}$, and therefore

$$\Phi_{i,j}^{(1,1)}(t,t') = \mathrm{e}^{-\Delta} I_{i-j}(\gamma\Delta)\left[\left(1-\frac{\gamma^2}{2}\right) - \frac{\gamma^2}{2}\mathcal{V}^{\boldsymbol{j}^{1,\mathrm{s}}}_{j-1,j+1}(t,t')\right].$$

Since both indices of $\mathcal{V}$ are components of $\boldsymbol{j}^{1,\mathrm{s}}$ we have to use the $(a, b) = (j_\mu, j_\nu)$ case of (60) to express this function. Making the corresponding substitution in the above expression produces the result (73) given in section 6.

The next higher nontrivial response functions are those with $k + l = 4$. As a second example we consider the case $k = l = 2$ where $\boldsymbol{i} = (i_1, i_2)$ and $\boldsymbol{j} = (j_1, j_2)$. We restrict ourselves further to vectors $\boldsymbol{j}$ that satisfy $j_2 = j_1 + 1$, corresponding to a perturbation that couples to adjacent sites of the lattice. In this situation we have two index vectors $\boldsymbol{j}^{\nu}$ given by $\boldsymbol{j}^1 = (j_1 - 1, j_1, j_1 + 1, j_2)$ and $\boldsymbol{j}^2 = (j_1, j_2 - 1, j_2, j_2 + 1)$. From (68) we obtain the corresponding squashed versions $\boldsymbol{j}^{1,\mathrm{s}} = (j_1 - 1, j_1)$ and $\boldsymbol{j}^{2,\mathrm{s}} = (j_2, j_2 + 1)$ using $j_2 = j_1 + 1$. Their dimensions are obviously $l^{1,\mathrm{s}} = l^{2,\mathrm{s}} = 2$. Now, by substituting $k = l = 2$ in (71), explicitly writing out the summations over μ and ν and plugging in $l^{\nu,\mathrm{s}}$ we obtain

$$\Phi_{\boldsymbol{i},\boldsymbol{j}}^{(2,2)} = \mathrm{e}^{-\Delta} I_{i_1-j_1}(\gamma\Delta)\left[\left(1-\frac{\gamma^2}{2}\right)\sum_{\pi\in\mathcal{P}'(2)}(-1)^{\pi}\prod_{\lambda=1}^{2}\mathcal{V}'^{\boldsymbol{j}}_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)}}(t,t') + \frac{\gamma^2}{2}\sum_{\pi\in\mathcal{P}'(2)}(-1)^{\pi}\prod_{\lambda=1}^{2}\mathcal{V}'^{\boldsymbol{j}^{1,\mathrm{s}}}_{(\boldsymbol{i}\cup\boldsymbol{j}^{1,\mathrm{s}})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j}^{1,\mathrm{s}})_{\pi(2\lambda)}}(t,t')\right] + \mathrm{e}^{-\Delta} I_{i_2-j_1}(\gamma\Delta)\left[\left(1-\frac{\gamma^2}{2}\right)\sum_{\pi\in\mathcal{P}'(2)}(-1)^{\pi}\prod_{\lambda=1}^{2}\mathcal{V}'^{\boldsymbol{j}}_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)}}(t,t')\right.$$

$$+\frac{\gamma^2}{2}\sum_{\pi\in\mathcal{P}'(2)}(-1)^\pi\prod_{\lambda=1}^{2}\mathcal{V}'^{\boldsymbol{j}^{1,\mathrm{s}}}_{(\boldsymbol{i}\cup\boldsymbol{j}^{1,\mathrm{s}})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j}^{1,\mathrm{s}})_{\pi(2\lambda)}}(t,t')\Bigg]$$

$$-\mathrm{e}^{-\Delta}I_{i_1-j_2}(\gamma\Delta)\Bigg[\left(1-\frac{\gamma^2}{2}\right)\sum_{\pi\in\mathcal{P}'(2)}(-1)^\pi\prod_{\lambda=1}^{2}\mathcal{V}'^{\boldsymbol{j}}_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)}}(t,t')$$

$$+\frac{\gamma^2}{2}\sum_{\pi\in\mathcal{P}'(2)}(-1)^\pi\prod_{\lambda=1}^{2}\mathcal{V}'^{\boldsymbol{j}^{2,\mathrm{s}}}_{(\boldsymbol{i}\cup\boldsymbol{j}^{2,\mathrm{s}})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j}^{2,\mathrm{s}})_{\pi(2\lambda)}}(t,t')\Bigg]$$

$$-\mathrm{e}^{-\Delta}I_{i_2-j_2}(\gamma\Delta)\Bigg[\left(1-\frac{\gamma^2}{2}\right)\sum_{\pi\in\mathcal{P}'(2)}(-1)^\pi\prod_{\lambda=1}^{2}\mathcal{V}'^{\boldsymbol{j}}_{(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j})_{\pi(2\lambda)}}(t,t')$$

$$+\frac{\gamma^2}{2}\sum_{\pi\in\mathcal{P}'(2)}(-1)^\pi\prod_{\lambda=1}^{2}\mathcal{V}'^{\boldsymbol{j}^{2,\mathrm{s}}}_{(\boldsymbol{i}\cup\boldsymbol{j}^{2,\mathrm{s}})_{\pi(2\lambda-1)},(\boldsymbol{i}\cup\boldsymbol{j}^{2,\mathrm{s}})_{\pi(2\lambda)}}(t,t')\Bigg].$$

Within the four square brackets the pairings are restricted to those, $\mathcal{P}'$, that contain the pairs (i_1, j_1), (i_2, j_1), (i_1, j_2) and (i_2, j_2) from top to bottom, respectively. The functions $\mathcal{V}'$ carrying these indices are to be replaced by 1. In all pairing sums the sets $\mathcal{P}'(2)$ are of course subsets of $\mathcal{P}(2)$; the latter contains the pairings (1, 2, 3, 4), (1, 3, 2, 4) and (1, 4, 2, 3). We now focus on the first square bracket; the remaining ones follow similarly. In the first sum, drawing pairs from $\boldsymbol{i}\cup\boldsymbol{j}=(i_1, i_2, j_1, j_2)$, the only pairing containing (i_1, j_1) is (1, 3, 2, 4) and has the sign -1. So this sum reduces to $-\mathcal{V}'_{i_1,j_1}\mathcal{V}'_{i_2,j_2}=-\mathcal{V}_{i_2,j_2}$. The second sum, on the other hand, draws pairs from $\boldsymbol{i}\cup\boldsymbol{j}^{1,\mathrm{s}}=(i_1, i_2, j_1-1, j_1)$. Hence the only pairing that contains (i_1, j_1) is (1, 4, 2, 3) having the sign +1. This sum therefore collapses to $+\mathcal{V}'_{i_1,j_1}\mathcal{V}'_{i_2,j_1-1}=+\mathcal{V}_{i_2,j_1-1}$. Applying this reasoning to all summations yields

$$\Phi^{(2,2)}_{\boldsymbol{i},\boldsymbol{j}}=\mathrm{e}^{-\Delta}I_{i_1-j_1}(\gamma\Delta)\left[-\left(1-\frac{\gamma^2}{2}\right)\mathcal{V}^{\boldsymbol{j}}_{i_2,j_2}(t,t')+\frac{\gamma^2}{2}\mathcal{V}^{\boldsymbol{j}^{1,\mathrm{s}}}_{i_2,j_1-1}(t,t')\right]$$

$$+\mathrm{e}^{-\Delta}I_{i_2-j_1}(\gamma\Delta)\left[+\left(1-\frac{\gamma^2}{2}\right)\mathcal{V}^{\boldsymbol{j}}_{i_1,j_2}(t,t')-\frac{\gamma^2}{2}\mathcal{V}^{\boldsymbol{j}^{1,\mathrm{s}}}_{i_1,j_1-1}(t,t')\right]$$

$$-\mathrm{e}^{-\Delta}I_{i_1-j_2}(\gamma\Delta)\left[+\left(1-\frac{\gamma^2}{2}\right)\mathcal{V}^{\boldsymbol{j}}_{i_2,j_1}(t,t')-\frac{\gamma^2}{2}\mathcal{V}^{\boldsymbol{j}^{2,\mathrm{s}}}_{i_2,j_2+1}(t,t')\right]$$

$$-\mathrm{e}^{-\Delta}I_{i_2-j_2}(\gamma\Delta)\left[-\left(1-\frac{\gamma^2}{2}\right)\mathcal{V}^{\boldsymbol{j}}_{i_1,j_1}(t,t')+\frac{\gamma^2}{2}\mathcal{V}^{\boldsymbol{j}^{2,\mathrm{s}}}_{i_1,j_2+1}(t,t')\right].$$

Note that the indices of all functions $\mathcal{V}$ comprise components of $\boldsymbol{i}$ and $\boldsymbol{j}$ or $\boldsymbol{i}$ and $\boldsymbol{j}^{\nu,\mathrm{s}}$. So all of them must be expressed via the $(a, b)=(i_\varepsilon, j_\nu)$ case of (60). We remind the reader that for ν in $(-1)^{\nu-1}$ in (60) one has to substitute the *actual component number* of the index $j_\nu=(\boldsymbol{j})_\nu$ in $\boldsymbol{j}$. This means, for instance, $j_1-1=(\boldsymbol{j}^{1,\mathrm{s}})_1$ is component $\nu=1$ in $\boldsymbol{j}^{1,\mathrm{s}}$ while $j_1=(\boldsymbol{j}^{1,\mathrm{s}})_2$ is component $\nu=2$ in $\boldsymbol{j}^{1,\mathrm{s}}$. Accurately applying (60) to the above expression yields the result (74) given in section 6.

References

[1] Glauber R J 1963 *J. Math. Phys.* **4** 294
[2] Bedeaux D, Shuler K E and Oppenheim I 1970 *J. Stat. Phys.* **2** 1
[3] Brey J J and Prados A 1993 *Physica* A **197** 569
[4] Brey J J and Prados A 1996 *Phys. Rev.* E **53** 458

[5] Prados A, Brey J J and Sánchez-Rey B 1997 *Europhys. Lett.* **40** 13
[6] Spouge J L 1988 *Phys. Rev. Lett.* **60** 871
[7] Amar J G and Family F 1990 *Phys. Rev.* A **41** 3258
[8] Santos J E 1997 *J. Phys. A: Math. Gen.* **30** 3249
[9] Brey J J and Prados A 2002 *Europhys. Lett.* **57** 171
[10] Godrèche C and Luck J M 2000 *J. Phys. A: Math. Gen.* **33** 1151
[11] Lippiello E and Zannetti M 2000 *Phys. Rev.* E **61** 3369
[12] Mayer P, Berthier L, Garrahan J P and Sollich P 2003 *Phys. Rev.* E **68** 016116
[13] Felderhof B U 1971 *Rep. Math. Phys.* **1** 215
[14] Aliev M A 1998 *Phys. Lett.* A **241** 19
[15] Derrida B and Zeitak R 1996 *Phys. Rev.* E **54** 2513
[16] Mayer P *PhD Thesis* King's College London (unpublished)
[17] Fontes L R, Isopi M, Newman C M and Stein D L 2001 *Phys. Rev. Lett.* **87** 110201
[18] Cugliandolo L F and Kurchan J 1993 *Phys. Rev. Lett.* **71** 173
[19] Cugliandolo L F, Kurchan J and Peliti L 1997 *Phys. Rev.* E **55** 3898
[20] Crisanti A and Ritort F 2003 *J. Phys. A: Math. Gen.* **36** 181
[21] Sollich P, Fielding S and Mayer P 2002 *J. Phys.: Condens. Matt.* **14** 1683
[22] Mayer P, Berthier L, Garrahan J P and Sollich P (unpublished)
[23] Krebs K, Pfannmüller M P, Wehefritz B and Hinrichsen H 1995 *J. Stat. Phys.* **78** 1429
[24] Fredrickson G H and Andersen H C 1984 *Phys. Rev. Lett.* **53** 1244
[25] Buhot A and Garrahan J P 2002 *Phys. Rev. Lett.* **88** 225702
[26] Gradshteyn L S and Ryzhik I M 2000 *Table of Integrals, Series, and Products* (New York: Academic)